Theoretical Informatics and Applications Will be set by the publisher
Informatique Théorique et Applications

RANDOM WALK IN A N-CUBE WITHOUT HAMILTONIAN CYCLE TO CHAOTIC PSEUDORANDOM NUMBER GENERATION: THEORETICAL AND PRACTICAL CONSIDERATIONS

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Abstract. This paper is dedicated to the design of chaotic random generators and extends previous works proposed by some of the authors. We propose a theoretical framework proving both the chaotic properties and that the limit distribution is uniform. A theoretical bound on the stationary time is given and practical experiments show that the generators successfully pass the classical statistical tests.

1991 Mathematics Subject Classification. 34C28, 37A25,11K45.

1. Introduction

The exploitation of chaotic systems to generate pseudorandom sequences is an hot topic [SPK01, SK01, CMZ09]. Such systems are fundamentally chosen due to their unpredictable character and their sensitiveness to initial conditions. In most cases, these generators simply consist in iterating a chaotic function like the logistic map [SPK01,SK01] or the Arnold's one [CMZ09]... It thus remains to find optimal parameters in such functions so that attractors are avoided, hoping by doing so that the generated numbers follow a uniform distribution. In order to check the quality of the produced outputs, it is usual to test the PRNGs (Pseudo-Random Number Generators) with statistical batteries like the so-called DieHARD [Mar96], NIST [BR10], or TestU01 [LS07] ones.

Keywords and phrases: Pseudorandom Number Generator, Theory of Chaos, Markov Matrice, Hamiltonian Path, Stopping Time, Statistical Test

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In its general understanding, chaos notion is often reduced to the strong sensitiveness to the initial conditions (the well known "butterfly effect"): a continuous function k defined on a metrical space is said *strongly sensitive to the initial conditions* if for each point x and each positive value ϵ, it is possible to find another point y as close as possible to x, and an integer t such that the distance between the t-th iterates of x and y, denoted by $k^t(x)$ and $k^t(y)$, are larger than ϵ. However, in his definition of chaos, Devaney [Dev89] imposes to the chaotic function two other properties called *transitivity* and *regularity*. Functions evoked above have been studied according to these properties, and they have been proven as chaotic on $\mathbb{R}$. But nothing guarantees that such properties are preserved when iterating the functions on floating point numbers, which is the domain of interpretation of real numbers $\mathbb{R}$ on machines.

To avoid this lack of chaos, we have previously presented some PRNGs that iterate continuous functions G_f on a discrete domain $\{1,\ldots,n\}^{\mathbb{N}} \times \{0,1\}^n$, where f is a Boolean function (*i.e.*, $f : \{0,1\}^{\mathsf{N}} \rightarrow \{0,1\}^{\mathsf{N}}$). These generators are $\mathit{CIPRNG}_f^1(u)$ [GWB10, BCGR11], $\mathit{CIPRNG}_f^2(u,v)$ [WBGF10] and $\chi_{\mathit{14Secrypt}}$ [CHG$^+$14] where *CI* means *Chaotic Iterations*. We have firstly proven in [BCGR11] that, to establish the chaotic nature of algorithm CIPRNG_f^1, it is necessary and sufficient that the asynchronous iterations are strongly connected. We then have proven that it is necessary and sufficient that the Markov matrix associated to this graph is doubly stochastic, in order to have a uniform distribution of the outputs. We have finally established sufficient conditions to guarantee the first property of connectivity. Among the generated functions, we thus have considered for further investigations only the ones that satisfy the second property too.

However, it cannot be directly deduced that $\chi_{\mathit{14Secrypt}}$ is chaotic since we do not output all the successive values of iterating G_f. This algorithm only displays a subsequence $x^{b.n}$ of a whole chaotic sequence x^n and it is indeed not correct that the chaos property is preserved for any subsequence of a chaotic sequence. This article presents conditions to preserve this property.

Finding a Boolean function which provides a strongly connected iteration graph having a doubly stochastic Markov matrix is however not an easy task. We have firstly proposed in [BCGR11] a generate-and-test based approach that solves this issue. However, this one was not efficient enough. Thus, a second approach has been further presented in [CHG$^+$14] by remarking that a N-cube where an Hamiltonian cycle (or equivalently a Gray code) has been removed is strongly connected and has a doubly stochastic Markov matrix.

However, the removed Hamiltonian cycle has a great influence in the quality of the output. For instance, if this one one is not balanced (*i.e.*, the number of changes in different bits are completely different), some bit would be hard to switch. This article shows an effective algorithm to provide functions issued from removing balanced Hamiltonian cycle in the N-cube.

The length b of the walk to reach a distribution close to the uniform one would be dramatically long. This article theoretically and practically studies the length b

until the corresponding Markov chain is close to the uniform distribution. Finally, the ability of the approach to face classical tests suite is evaluated.

The remainder of this article is organized as follows. The next section is devoted to preliminaries, basic notations, and terminologies regarding Boolean map iterations. Then, in Section 3, Devaney's definition of chaos is recalled while the proofs of chaos of our most general PRNGs is provided. This is the first major contribution. Section 4 recalls a general scheme to obtain functions with awaited behavior. Main theorems are recalled to make the document self-content. The next section (Sect. 5) presents an algorithm that implements this scheme and proves it always produces a solution. This is the second major contribution The later section (Sect. 6) defines the theoretical framework to study the mixing-time, *i.e.*, time until reaching a uniform distribution. It proves that this one is at worth quadratic in the number of elements. Experiments show that the bound is practically largely much lower. This is the third major contribution. The Section 7 gives practical results on evaluating the PRNG against the NIST suite. This research work ends by a conclusion section, where the contribution is summarized and intended future work is outlined.

2. Preliminaries

In what follows, we consider the Boolean algebra on the set $\mathbb{B} = \{0, 1\}$ with the classical operators of conjunction '.', of disjunction '+', of negation '-', and of disjunctive union $\oplus$.

Let us first introduce basic notations. Let N be a positive integer. The set $\{1, 2, \dots, \mathsf{N}\}$ of integers belonging between 1 and N is further denoted as $[\![1, \mathsf{N}]\!]$. A *Boolean map* f is a function from $\mathbb{B}^{\mathsf{N}}$ to itself such that $x = (x_1, \dots, x_{\mathsf{N}})$ maps to $f(x) = (f_1(x), \dots, f_{\mathsf{N}}(x))$. In what follows, for any finite set X, $|X|$ denotes its cardinality and $\lfloor y \rfloor$ is the largest integer lower than y.

Functions are iterated as follows. At the t^{th} iteration, only the s_t-th component is said to be "iterated", where $s = (s_t)_{t \in \mathbb{N}}$ is a sequence of indices taken in $[\![1; \mathsf{N}]\!]$ called "strategy". Formally, let $F_f : \mathbb{B}^{\mathsf{N}} \times [\![1; \mathsf{N}]\!]$ to $\mathbb{B}^{\mathsf{N}}$ be defined by

$$F_f(x, i) = (x_1, \dots, x_{i-1}, f_i(x), x_{i+1}, \dots, x_{\mathsf{N}}).$$

Then, let $x^0 \in \mathbb{B}^{\mathsf{N}}$ be an initial configuration and $s \in [\![1; \mathsf{N}]\!]^{\mathbb{N}}$ be a strategy, the dynamics are described by the recurrence

$$x^{t+1} = F_f(x^t, s_t). \tag{1}$$

Let be given a Boolean map f. Its associated *iteration graph* $\Gamma(f)$ is the directed graph such that the set of vertices is $\mathbb{B}^{\mathsf{N}}$, and for all $x \in \mathbb{B}^{\mathsf{N}}$ and $i \in [\![1; \mathsf{N}]\!]$, the graph $\Gamma(f)$ contains an arc from x to $F_f(x, i)$. Each arc $(x, F_f(x, i))$ is labelled with i.

Running Example. *Let us consider for instance* $\mathsf{N} = 3$. *Let* $f^* : \mathbb{B}^3 \to \mathbb{B}^3$ *be defined by* $f^*(x_1, x_2, x_3) = (x_2 \oplus x_3, \overline{x_1}\overline{x_3} + x_1\overline{x_2}, \overline{x_1}\overline{x_3} + x_1x_2)$. *The iteration graph* $\Gamma(f^*)$ *of this function is given in Figure 1.*

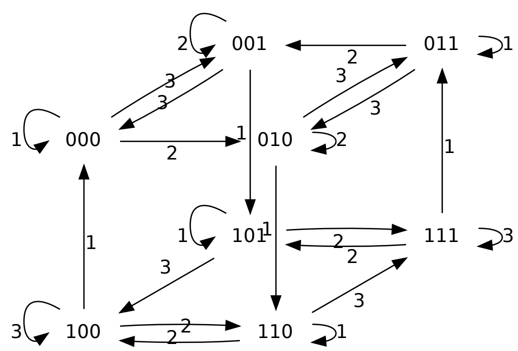


FIGURE 1. Iteration Graph $\Gamma(f^*)$ of the function f^*

Let us finally recall the pseudorandom number generator $\chi_{14Secrypt}$ [CHG+14] formalized in Algorithm 1. It is based on random walks in $\Gamma(f)$. More precisely, let be given a Boolean map $f : \mathbb{B}^\mathsf{N} \to \mathbb{B}^\mathsf{N}$, an input PRNG *Random*, an integer b that corresponds to a number of iterations, and an initial configuration x^0. Starting from x^0, the algorithm repeats b times a random choice of which edge to follow and traverses this edge. The final configuration is thus outputted.

Input: a function f, an iteration number b, an initial configuration x^0 (N bits)
Output: a configuration x (N bits)
$x \leftarrow x^0$;
for $i = 0, \dots, b-1$ **do**
  $s \leftarrow Random(\mathsf{N})$;
  $x \leftarrow F_f(x, s)$;
end
return x;

Algorithm 1: Pseudo Code of the $\chi_{14Secrypt}$ PRNG

Based on this setup, we can study the chaos properties of these function. This is the aims of the next section.

3. Proof Of Chaos

Let us us first recall the chaos theoretical context presented in [BCGR11]. In this article, the space of interest is $\mathbb{B}^\mathsf{N} \times [\![1; \mathsf{N}]\!]^\mathbb{N}$ and the iteration function $\mathcal{H}_f$ is the map from $\mathbb{B}^{\mathsf{N}} \times [\![1;\mathsf{N}]\!]^{\mathbb{N}}$ to itself defined by

$$\mathcal{H}_f(x,s) = (F_f(x,s_0), \sigma(s)).$$

In this definition, $\sigma : [\![1;\mathsf{N}]\!]^{\mathbb{N}} \longrightarrow [\![1;\mathsf{N}]\!]^{\mathbb{N}}$ is a shift operation on sequences (*i.e.*, a function that removes the first element of the sequence) formally defined with

$$\sigma((u^k)_{k\in\mathbb{N}}) = (u^{k+1})_{k\in\mathbb{N}}.$$

We have proven [BCGR11, Theorem 1] that $\mathcal{H}_f$ is chaotic in $\mathbb{B}^{\mathsf{N}} \times [\![1;\mathsf{N}]\!]^{\mathbb{N}}$ if and only if $\Gamma(f)$ is strongly connected. However, the corrolary which would say that $\chi_{\mathit{14Secrypt}}$ is chaotic cannot be directly deduced since we do not output all the successive values of iterating F_f. Only a a few of them is concerned and any subsequence of a chaotic sequence is not necessarily a chaotic sequence too. This necessitates a rigorous proof, which is the aim of this section.

3.1. Devaney's Chaotic Dynamical Systems

Consider a topological space $(\mathcal{X},\tau)$ and a continuous function $f : \mathcal{X} \to \mathcal{X}$.

Definition 3.1. The function f is said to be *topologically transitive* if, for any pair of open sets $U, V \subset \mathcal{X}$, there exists $k > 0$ such that $f^k(U) \cap V \neq \varnothing$.

Definition 3.2. An element x is a *periodic point* for f of period $n \in \mathbb{N}^*$ if $f^n(x) = x$.

Definition 3.3. f is said to be *regular* on $(\mathcal{X},\tau)$ if the set of periodic points for f is dense in $\mathcal{X}$: for any point x in $\mathcal{X}$, any neighborhood of x contains at least one periodic point (without necessarily the same period).

Definition 3.4 (Devaney's formulation of chaos [Dev89])**.** The function f is said to be *chaotic* on $(\mathcal{X},\tau)$ if f is regular and topologically transitive.

The chaos property is strongly linked to the notion of "sensitivity", defined on a metric space $(\mathcal{X},d)$ by:

Definition 3.5. The function f has *sensitive dependence on initial conditions* if there exists $\delta > 0$ such that, for any $x \in \mathcal{X}$ and any neighborhood V of x, there exist $y \in V$ and $n > 0$ such that $d(f^n(x), f^n(y)) > \delta$.

The constant δ is called the *constant of sensitivity* of f.

Indeed, Banks *et al.* have proven in [BBCS92] that when f is chaotic and $(\mathcal{X},d)$ is a metric space, then f has the property of sensitive dependence on initial conditions (this property was formerly an element of the definition of chaos).

3.2. A Metric Space for PRNG Iterations

Let us first introduce $\mathcal{P} \subset \mathbb{N}$ a finite nonempty set having the cardinality $\mathsf{p} \in \mathbb{N}^*$. Intuitively, this is the set of authorized numbers of iterations. Denote by

$p_1, p_2, \ldots, p_{\mathsf{p}}$ the ordered elements of $\mathcal{P}$: $\mathcal{P} = \{p_1, p_2, \ldots, p_{\mathsf{p}}\}$ and $p_1 < p_2 < \ldots < p_{\mathsf{p}}$. In our algorithm, p is 1 and p_1 is b.

The Algorithm 1 may be seen as b functional composition of F_f. However, it can be generalized with p_i, $p_i \in \mathcal{P}$, functional compositions of F_f. Thus, for any $p_i \in \mathcal{P}$ we introduce the function $F_{f,p_i} : \mathbb{B}^{\mathsf{N}} \times [\![1, \mathsf{N}]\!]^{p_i} \to \mathbb{B}^{\mathsf{N}}$ defined by

$$F_{f,p_i}(x, (u^0, u^1, \ldots, u^{p_i-1})) \mapsto F_f(\ldots(F_f(F_f(x, u^0), u^1), \ldots), u^{p_i-1}).$$

The considered space is $\mathcal{X}_{\mathsf{N},\mathcal{P}} = \mathbb{B}^{\mathsf{N}} \times \mathbb{S}_{\mathsf{N},\mathcal{P}}$, where $\mathbb{S}_{\mathsf{N},\mathcal{P}} = [\![1, \mathsf{N}]\!]^{\mathbb{N}} \times \mathcal{P}^{\mathbb{N}}$. Each element in this space is a pair where the first element is N-uple in $\mathbb{B}^{\mathsf{N}}$, as in the previous space. The second element is a pair $((u^k)_{k\in\mathbb{N}}, (v^k)_{k\in\mathbb{N}})$ of infinite sequences. The sequence $(v^k)_{k\in\mathbb{N}}$ defines how many iterations are executed at time k between two outputs. The sequence $(u^k)_{k\in\mathbb{N}}$ defines which elements is modified.

Let us define the shift function Σ for any element of $\mathbb{S}_{\mathsf{N},\mathcal{P}}$.

$$\begin{array}{rccc} \Sigma : & \mathbb{S}_{\mathsf{N},\mathcal{P}} & \longrightarrow & \mathbb{S}_{\mathsf{N},\mathcal{P}} \\ & \left((u^k)_{k\in\mathbb{N}}, (v^k)_{k\in\mathbb{N}}\right) & \longmapsto & \left(\sigma^{v^0}\left((u^k)_{k\in\mathbb{N}}\right), \sigma\left((v^k)_{k\in\mathbb{N}}\right)\right). \end{array}$$

In other words, Σ receives two sequences u and v, and it operates v^0 shifts on the first sequence and a single shift on the second one. Let

$$\begin{array}{rccc} G_f : & \mathcal{X}_{\mathsf{N},\mathcal{P}} & \longrightarrow & \mathcal{X}_{\mathsf{N},\mathcal{P}} \\ & (e, (u, v)) & \longmapsto & \left(F_{f,v^0}\left(e, (u^0, \ldots, u^{v^0-1}\right), \Sigma(u, v)\right). \end{array} \quad (2)$$

Then the outputs $(y^0, y^1, \ldots)$ produced by the $CIPRNG_f^2(u, v)$ generator are the first components of the iterations $X^0 = (x^0, (u, v))$ and $\forall n \in \mathbb{N}, X^{n+1} = G_f(X^n)$ on $\mathcal{X}_{\mathsf{N},\mathcal{P}}$.

3.3. A metric on $\mathcal{X}_{\mathsf{N},\mathcal{P}}$

We define a distance d on $\mathcal{X}_{\mathsf{N},\mathcal{P}}$ as follows. Consider $x = (e, s)$ and $\check{x} = (\check{e}, \check{s})$ in $\mathcal{X}_{\mathsf{N},\mathcal{P}} = \mathbb{B}^{\mathsf{N}} \times \mathbb{S}_{\mathsf{N},\mathcal{P}}$, where $s = (u, v)$ and $\check{s} = (\check{u}, \check{v})$ are in $\mathbb{S}_{\mathsf{N},\mathcal{P}} = \mathcal{S}_{[\![1,\mathsf{N}]\!]} \times \mathcal{S}_{\mathcal{P}}$.

- e and $\check{e}$ are integers belonging in $[\![0, 2^{\mathsf{N}-1}]\!]$. The Hamming distance on their binary decomposition, that is, the number of dissimilar binary digits, constitutes the integral part of $d(X, \check{X})$.
- The fractional part is constituted by the differences between v^0 and $\check{v}^0$, followed by the differences between finite sequences $u^0, u^1, \ldots, u^{v^0-1}$ and $\check{u}^0, \check{u}^1, \ldots, \check{u}^{\check{v}^0-1}$, followed by differences between v^1 and $\check{v}^1$, followed by the differences between $u^{v^0}, u^{v^0+1}, \ldots, u^{v^1-1}$ and $\check{u}^{\check{v}^0}, \check{u}^{\check{v}^0+1}, \ldots, \check{u}^{\check{v}^1-1}$, etc. More precisely, let $p = \lfloor \log_{10}(\max \mathcal{P}) \rfloor + 1$ and $n = \lfloor \log_{10}(\mathsf{N}) \rfloor + 1$.
  - The p first digits of $d(x, \check{x})$ is $|v^0 - \check{v}^0|$ written in decimal numeration (and with p digits).

- The next $n \times \max(\mathcal{P})$ digits aim at measuring how much $u^0, u^1, \ldots, u^{v^0-1}$ differs from $\check{u}^0, \check{u}^1, \ldots, \check{u}^{\check{v}^0-1}$. The n first digits are $|u^0 - \check{u}^0|$. They are followed by $|u^1 - \check{u}^1|$ written with n digits, etc.
  * If $v^0 = \check{v}^0$, then the process is continued until $|u^{v^0-1} - \check{u}^{\check{v}^0-1}|$ and the fractional part of $d(X, \check{X})$ is completed by 0's until reaching $p + n \times \max(\mathcal{P})$ digits.
  * If $v^0 < \check{v}^0$, then the $\max(\mathcal{P})$ blocs of n digits are $|u^0 - \check{u}^0|$, ..., $|u^{v^0-1} - \check{u}^{v^0-1}|$, $\check{u}^{v^0}$ (on n digits), ..., $\check{u}^{\check{v}^0-1}$ (on n digits), followed by 0's if required.
  * The case $v^0 > \check{v}^0$ is dealt similarly.
- The next p digits are $|v^1 - \check{v}^1|$, etc.

Running Example. *Consider for instance that* $\mathsf{N} = 13$, $\mathcal{P} = \{1, 2, 11\}$ *(so* $\mathsf{p} = 3$*), and that* $s = \begin{cases} u = \underline{6,}\ \underline{11,5,}\ldots \\ v = 1, 2, \ldots \end{cases}$ *while* $\check{s} = \begin{cases} \check{u} = \underline{6,4}\ \underline{1,}\ldots \\ \check{v} = 2, 1, \ldots \end{cases}$.

So $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(s, \check{s}) = 0.010004000000000000000000011005\ldots$ *Indeed, the* $p = 2$ *first digits are 01, as* $|v^0 - \check{v}^0| = 1$*, and we use* p *digits to code this difference (*$\mathcal{P}$ *being* $\{1, 2, 11\}$*, this difference can be equal to 10). We then take the* $v^0 = 1$ *first terms of* u*, each term being coded in* $n = 2$ *digits, that is, 06. As we can iterate at most* $\max(\mathcal{P})$ *times, we must complete this value by some 0's in such a way that the obtained result has* $n \times \max(\mathcal{P}) = 22$ *digits, that is: 0600000000000000000000. Similarly, the* $\check{v}^0 = 2$ *first terms in* $\check{u}$ *are represented by 0604000000000000000000, and the absolute value of their difference is equal to 0004000000000000000000. These digits are concatenated to 01, and we start again with the remainder of the sequences.*

Running Example. *Consider now that* $\mathsf{N} = 9$*, and* $\mathcal{P} = \{2, 7\}$*, and that* $s = \begin{cases} u = \underline{6,7,}\ \underline{4,2,}\ldots \\ v = 2, 2, \ldots \end{cases}$ *while* $\check{s} = \begin{cases} \check{u} = \underline{4,9,6,3,6,6,7,}\ \underline{9,8,}\ldots \\ \check{v} = 7, 2, \ldots \end{cases}$

So $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(s, \check{s}) = 0.5173633305600000\ldots$*, as* $|v^0 - \check{v}^0| = 5$, $|4963667 - 6700000| = 1736333$, $|v^1 - \check{v}^1| = 0$*, and* $|9800000 - 4200000| = 5600000$.

d can be more rigorously written as follows:

$$d(x, \check{x}) = d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(s, \check{s}) + d_{\mathbb{B}^{\mathsf{N}}}(e, \check{e}),$$

where:

- $d_{\mathbb{B}^{\mathsf{N}}}$ is the Hamming distance,
- $\forall s = (u, v), \check{s} = (\check{u}, \check{v}) \in \mathcal{S}_{\mathsf{N},\mathcal{P}}$,

$$\begin{aligned} d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(s, \check{s}) &= \sum_{k=0}^{\infty} \frac{1}{10^{(k+1)p + kn\max(\mathcal{P})}} \Bigg(|v^k - \check{v}^k| \\ &\quad + \left| \sum_{l=0}^{v^k-1} \frac{u^{\sum_{m=0}^{k-1} v^m + l}}{10^{(l+1)n}} - \sum_{l=0}^{\check{v}^k-1} \frac{\check{u}^{\sum_{m=0}^{k-1} \check{v}^m + l}}{10^{(l+1)n}} \right| \Bigg) \end{aligned}$$

Let us show that,

Proposition 3.6. *d is a distance on $\mathcal{X}_{\mathsf{N},\mathcal{P}}$.*

Proof. $d_{\mathbb{B}^\mathsf{N}}$ is the Hamming distance. We will prove that $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}$ is a distance too, thus d will also be a distance, being the sum of two distances.

- Obviously, $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(s,\check{s}) \geqslant 0$, and if $s = \check{s}$, then $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(s,\check{s}) = 0$. Conversely, if $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(s,\check{s}) = 0$, then $\forall k \in \mathbb{N}, v^k = \check{v}^k$ due to the definition of d. Then, as digits between positions $p+1$ and $p+n$ are null and correspond to $|u^0 - \check{u}^0|$, we can conclude that $u^0 = \check{u}^0$. An extension of this result to the whole first $n \times \max(\mathcal{P})$ bloc leads to $u^i = \check{u}^i$, $\forall i \leqslant v^0 = \check{v}^0$, and by checking all the $n \times \max(\mathcal{P})$ blocs, $u = \check{u}$.
- $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}$ is clearly symmetric ($d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(s,\check{s}) = d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(\check{s},s)$).
- The triangle inequality is obtained because the absolute value satisfies it too.

□

Before being able to study the topological behavior of the general chaotic iterations, we must first establish that:

Proposition 3.7. *For all $f : \mathbb{B}^\mathsf{N} \longrightarrow \mathbb{B}^\mathsf{N}$, the function G_f is continuous on $(\mathcal{X}, d)$.*

Proof. We will show this result by using the sequential continuity. Consider a sequence $x^n = (e^n, (u^n, v^n)) \in \mathcal{X}_{\mathsf{N},\mathcal{P}}^{\mathbb{N}}$ such that $d(x^n, x) \longrightarrow 0$, for some $x = (e, (u, v)) \in \mathcal{X}_{\mathsf{N},\mathcal{P}}$. We will show that $d(G_f(x^n), G_f(x)) \longrightarrow 0$. Remark that u and v are sequences of sequences.

As $d(x^n, x) \longrightarrow 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x^n, x) < 10^{-(p+n\max(\mathcal{P}))}$ (its $p + n\max(\mathcal{P})$ first digits are null). In particular, $\forall n \geqslant n_0, e^n = e$, as the Hamming distance between the integral parts of x and $\check{x}$ is 0. Similarly, due to the nullity of the $p + n\max(\mathcal{P})$ first digits of $d(x^n, x)$, we can conclude that $\forall n \geqslant n_0$, $(v^n)^0 = v^0$, and that $\forall n \geqslant n_0$, $(u^n)^0 = u^0$, $(u^n)^1 = u^1$, ..., $(u^n)^{v^0-1} = u^{v^0-1}$. This implies that:

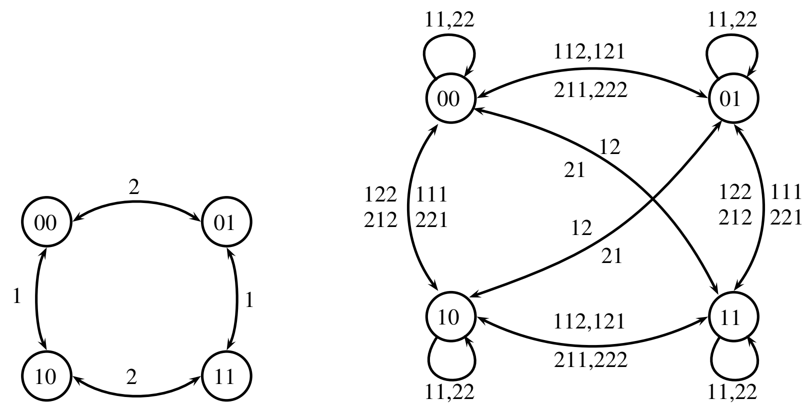
- $G_f(x^n)_1 = G_f(x)_1$: they have the same Boolean vector as first coordinate.
- $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}(\Sigma(u^n, v^n); \Sigma(u, v)) = 10^{p+n\max(\mathcal{P})} d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}((u^n, v^n); (u, v))$. As the right part of the equality tends to 0, we can deduce that it is the case too for the left part of the equality, and so $G_f(x^n)_2$ is convergent to $G_f(x)_2$.

□

3.4. $\Gamma_\mathcal{P}(f)$ as an extension of $\Gamma(f)$

Let $\mathcal{P} = \{p_1, p_2, \ldots, p_\mathsf{p}\}$. We define the directed graph $\Gamma_\mathcal{P}(f)$ as follows.

- Its vertices are the 2^N elements of $\mathbb{B}^\mathsf{N}$.
- Each vertex has $\displaystyle\sum_{i=1}^{\mathsf{p}} \mathsf{N}^{p_i}$ arrows, namely all the $p_1, p_2, \ldots, p_\mathsf{p}$ tuples having their elements in $[\![1, \mathsf{N}]\!]$.
- There is an arc labeled $u_0, \ldots, u_{p_i-1}$, $i \in [\![1, \mathsf{p}]\!]$ between vertices x and y if and only if $y = F_{f,p_i}(x, (u_0, \ldots, u_{p_i-1}))$.



(A) $\Gamma(f_0)$ (B) $\Gamma_{\{2,3\}}(f_0)$

FIGURE 2. Iterating $f_0 : (x_1, x_2) \mapsto (\overline{x_1}, \overline{x_2})$

It is not hard to see that the graph $\Gamma_{\{1\}}(f)$ is $\Gamma(f)$.

Running Example. *Consider for instance* $\mathsf{N} = 2$*, Let* $f_0 : \mathbb{B}^2 \longrightarrow \mathbb{B}^2$ *be the negation function, i.e.,* $f_0(x_1, x_2) = (\overline{x_1}, \overline{x_2})$*, and consider* $\mathcal{P} = \{2, 3\}$*. The graphs of iterations are given in* FIGURE 2*. The* FIGURE 2A *shows what happens when displaying each iteration result. On the contrary, the* FIGURE 2B *explicits the behaviors when always applying 2 or 3 modification and next outputing results. Notice that here, orientations of arcs are not necessary since the function* f_0 *is equal to its inverse* f_0^{-1}*.*

3.5. PROOFS OF CHAOS

We will show that,

Proposition 3.8. $\Gamma_{\mathcal{P}}(f)$ *is strongly connected if and only if* G_f *is topologically transitive on* $(\mathcal{X}_{\mathsf{N},\mathcal{P}}, d)$*.*

Proof. Suppose that $\Gamma_{\mathcal{P}}(f)$ is strongly connected. Let $x = (e, (u, v)), \check{x} = (\check{e}, (\check{u}, \check{v})) \in \mathcal{X}_{\mathsf{N},\mathcal{P}}$ and $\varepsilon > 0$. We will find a point y in the open ball $\mathcal{B}(x, \varepsilon)$ and $n_0 \in \mathbb{N}$ such that $G_f^{n_0}(y) = \check{x}$: this strong transitivity will imply the transitivity property. We can suppose that $\varepsilon < 1$ without loss of generality.

Let us denote by $(E, (U, V))$ the elements of y. As y must be in $\mathcal{B}(x, \varepsilon)$ and $\varepsilon < 1$, E must be equal to e. Let $k = \lfloor \log_{10}(\varepsilon) \rfloor + 1$. $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}((u, v), (U, V))$ must be lower than ε, so the k first digits of the fractional part of $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}((u, v), (U, V))$ are null. Let k_1 the smallest integer such that, if $V^0 = v^0$, ..., $V^{k_1} = v^{k_1}$, $U^0 = u^0$, ..., $U^{\sum_{l=0}^{k_1} V^l - 1} = u^{\sum_{l=0}^{k_1} v^l - 1}$. Then $d_{\mathbb{S}_{\mathsf{N},\mathcal{P}}}((u, v), (U, V)) < \varepsilon$. In other words, any y of the form $(e, ((u^0, ..., u^{\sum_{l=0}^{k_1} v^l - 1}), (v^0, ..., v^{k_1})))$ is in $\mathcal{B}(x, \varepsilon)$.

Let y^0 such a point and $z = G_f^{k_1}(y^0) = (e', (u', v'))$. $\Gamma_{\mathcal{P}}(f)$ being strongly connected, there is a path between e' and $\check{e}$. Denote by $a_0, \ldots, a_{k_2}$ the edges visited by this path. We denote by $V^{k_1} = |a_0|$ (number of terms in the finite sequence a_1), $V^{k_1+1} = |a_1|$, ..., $V^{k_1+k_2} = |a_{k_2}|$, and by $U^{k_1} = a_0^0$, $U^{k_1+1} = a_0^1$, ..., $U^{k_1+V_{k_1}-1} = a_0^{V_{k_1}-1}$, $U^{k_1+V_{k_1}} = a_1^0$, $U^{k_1+V_{k_1}+1} = a_1^1$,...

Let $y = (e, ((u^0, ..., u^{\sum_{l=0}^{k_1} v^l - 1}, a_0^0, ..., a_0^{|a_0|}, a_1^0, ..., a_1^{|a_1|}, ..., a_{k_2}^0, ..., a_{k_2}^{|a_{k_2}|}, \check{u}^0, \check{u}^1, ...), (v^0, ..., v^{k_1}, |a_0|, ..., |a_{k_2}|, \check{v}^0, \check{v}^1, ...)))$. So $y \in \mathcal{B}(x, \varepsilon)$ and $G_f^{k_1+k_2}(y) = \check{x}$.

Conversely, if $\Gamma_{\mathcal{P}}(f)$ is not strongly connected, then there are 2 vertices e_1 and e_2 such that there is no path between e_1 and e_2. That is, it is impossible to find $(u, v) \in \mathbb{S}_{\mathsf{N},\mathcal{P}}$ and $n\mathbb{N}$ such that $G_f^n(e, (u, v))_1 = e_2$. The open ball $\mathcal{B}(e_2, 1/2)$ cannot be reached from any neighborhood of e_1, and thus G_f is not transitive. □

We show now that,

Proposition 3.9. *If $\Gamma_{\mathcal{P}}(f)$ is strongly connected, then G_f is regular on $(\mathcal{X}_{\mathsf{N},\mathcal{P}}, d)$.*

Proof. Let $x = (e, (u, v)) \in \mathcal{X}_{\mathsf{N},\mathcal{P}}$ and $\varepsilon > 0$. As in the proofs of Prop. 3.8, let $k_1 \in \mathbb{N}$ such that

$$\Big\{(e, ((u^0, ..., u^{v^{k_1-1}}, U^0, U^1, ...), (v^0, ..., v^{k_1}, V^0, V^1, ...)) \mid$$

$$\forall i, j \in \mathbb{N}, U^i \in [\![1, \mathsf{N}]\!], V^j \in \mathcal{P}\Big\} \subset \mathcal{B}(x, \varepsilon),$$

and $y = G_f^{k_1}(e, (u, v))$. $\Gamma_{\mathcal{P}}(f)$ being strongly connected, there is at least a path from the Boolean state y_1 of y and e. Denote by $a_0, \ldots, a_{k_2}$ the edges of such a path. Then the point:

$$(e, ((u^0, ..., u^{v^{k_1-1}}, a_0^0, ..., a_0^{|a_0|}, a_1^0, ..., a_1^{|a_1|}, ..., a_{k_2}^0, ..., a_{k_2}^{|a_{k_2}|}, u^0, ..., u^{v^{k_1-1}},$$

$$a_0^0, ..., a_{k_2}^{|a_{k_2}|} ...), (v^0, ..., v^{k_1}, |a_0|, ..., |a_{k_2}|, v^0, ..., v^{k_1}, |a_0|, ..., |a_{k_2}|, ...))$$

is a periodic point in the neighborhood $\mathcal{B}(x, \varepsilon)$ of x. □

G_f being topologically transitive and regular, we can thus conclude that

Theorem 3.10. *The function G_f is chaotic on $(\mathcal{X}_{\mathsf{N},\mathcal{P}}, d)$ if and only if its iteration graph $\Gamma_{\mathcal{P}}(f)$ is strongly connected.*

Corollary 3.11. *The pseudorandom number generator $\chi_{14Secrypt}$ is not chaotic on $(\mathcal{X}_{\mathsf{N},\{b\}}, d)$ for the negation function.*

Proof. In this context, $\mathcal{P}$ is the singleton $\{b\}$. If b is even, any vertex e of $\Gamma_{\{b\}}(f_0)$ cannot reach its neighborhood and thus $\Gamma_{\{b\}}(f_0)$ is not strongly connected. If b is even, any vertex e of $\Gamma_{\{b\}}(f_0)$ cannot reach itself and thus $\Gamma_{\{b\}}(f_0)$ is not strongly connected. □

The next section shows how to generate functions and a iteration number b such that $\Gamma_{\{b\}}$ is strongly connected.

4. Functions with Strongly Connected $\Gamma_{\{b\}}(f)$

First of all, let $f : \mathbb{B}^{\mathsf{N}} \rightarrow \mathbb{B}^{\mathsf{N}}$. It has been shown [BCGR11, Theorem 4] that if its iteration graph $\Gamma(f)$ is strongly connected, then the output of $\chi_{14Secrypt}$ follows a law that tends to the uniform distribution if and only if its Markov matrix is a doubly stochastic matrix.

In [CHG+14, Section 4], we have presented an efficient approach which generates function with strongly connected iteration graph $\Gamma(f)$ and with doubly stochastic Markov probability matrix.

Basically, let us consider the N-cube. Let us next remove one Hamiltonian cycle in this one. When an edge (x, y) is removed, an edge (x, x) is added.

Running Example. *For instance, the iteration graph $\Gamma(f^*)$ (given in Figure 1) is the 3-cube in which the Hamiltonian cycle $000, 100, 101, 001, 011, 111, 110, 010, 000$ has been removed.*

We first have proven the following result, which states that the N-cube without one Hamiltonian cycle has the awaited property with regard to the connectivity.

Theorem 4.1. *The iteration graph $\Gamma(f)$ issued from the N-cube where an Hamiltonian cycle is removed is strongly connected.*

Moreover, if all the transitions have the same probability ($\frac{1}{n}$), we have proven the following results:

Theorem 4.2. *The Markov Matrix M resulting from the N-cube in which an Hamiltonian cycle is removed, is doubly stochastic.*

Let us consider now a N-cube where an Hamiltonian cycle is removed. Let f be the corresponding function. The question which remains to solve is can we always find b such that $\Gamma_{\{b\}}(f)$ is strongly connected.

The answer is indeed positive. We furtheremore have the following strongest result.

Theorem 4.3. *There exist $b \in \mathbb{N}$ such that $\Gamma_{\{b\}}(f)$ is complete.*

Proof. There is an arc (x, y) in the graph $\Gamma_{\{b\}}(f)$ if and only if M^b_{xy} is positive where M is the Markov matrix of $\Gamma(f)$. It has been shown in [BCGR11, Lemma 3] that M is regular. There exists thus b such there is an arc between any x and y. □

The next section presents how to build hamiltonian cycles in the N-cube with the objective to embed them into the pseudorandom number generator.

5. Balanced Hamiltonian Cycle

Many approaches have been developed to solve the problem of building a Gray code in a N cube [RC81, BS96, SZ04, Byk16], according to properties the produced code has to verify. For instance, [BS96, SZ04] focus on balanced Gray codes. In the transition sequence of these codes, the number of transitions of each element must differ at most by 2. This uniformity is a global property on the cycle, *i.e.* a property that is established while traversing the whole cycle. On the opposite side, when the objective is to follow a subpart of the Gray code and to switch each element approximately the same amount of times, local properties are wished. For instance, the locally balanced property is studied in [Byk16] and an algorithm that establishes locally balanced Gray codes is given.

The current context is to provide a function $f : \mathbb{B}^{\mathsf{N}} \rightarrow \mathbb{B}^{\mathsf{N}}$ by removing a Hamiltonian cycle in the N cube. Such a function is going to be iterated b times to produce a pseudo random number, *i.e.* a vertex in the N cube. Obviously, the number of iterations b has to be sufficiently large to provide a uniform output distribution. To reduce the number of iterations, the provided Gray code should ideally possess the both balanced and locally balanced properties. However, none of the two algorithms is compatible with the second one: balanced Gray codes that are generated by state of the art works [SZ04, BS96] are not locally balanced. Conversely, locally balanced Gray codes yielded by Igor Bykov approach [Byk16] are not globally balanced. This section thus shows how the non deterministic approach presented in [SZ04] has been automatized to provide balanced Hamiltonian paths such that, for each subpart, the number of switches of each element is as uniform as possible.

5.1. Analysis of the Robinson-Cohn extension algorithm

As far as we know three works, namely [RC81], [BS96], and [SZ04] have addressed the problem of providing an approach to produce balanced gray code. The authors of [RC81] introduced an inductive approach aiming at producing balanced Gray codes, provided the user gives a special subsequence of the transition sequence at each induction step. This work have been strengthened in [BS96] where the authors have explicitly shown how to construct such a subsequence. Finally the authors of [SZ04] have presented the *Robinson-Cohn extension* algorithm. There rigorous presentation of this one have mainly allowed them to prove two properties. The former states that if N is a 2-power, a balanced Gray code is always totally balanced. The latter states that for every N there exists a Gray code such that all transition count numbers are are 2-powers whose exponents are either equal or differ from each other by 1. However, the authors do not prove that the approach allows to build (totally balanced) Gray code. What follows shows that this fact is established and first recalls the approach.

Let be given a $\mathsf{N}-2$-bit Gray code whose transition sequence is $S_{\mathsf{N}-2}$. What follows is the *Robinson-Cohn extension* method [SZ04] which produces a n-bits Gray code.

(1) Let l be an even positive integer. Find $u_1, u_2, \dots, u_{l-2}, v$ (maybe empty) subsequences of $S_{\mathsf{N}-2}$ such that $S_{\mathsf{N}-2}$ is the concatenation of

$$s_{i_1}, u_0, s_{i_2}, u_1, s_{i_3}, u_2, \dots, s_{i_l-1}, u_{l-2}, s_{i_l}, v$$

where $i_1 = 1$, $i_2 = 2$, and $u_0 = \emptyset$ (the empty sequence).

(2) Replace in $S_{\mathsf{N}-2}$ the sequences $u_0, u_1, u_2, \ldots, u_{l-2}$ by $\mathsf{N}-1, u'(u_1, \mathsf{N}-1, \mathsf{N}), u'(u_2, \mathsf{N}, \mathsf{N}-1), u'(u_3, \mathsf{N}-1, \mathsf{N}), \ldots, u'(u_{l-2}, \mathsf{N}, \mathsf{N}-1)$ respectively, where $u'(u, x, y)$ is the sequence u, x, u^R, y, u such that u^R is u in reversed order. The obtained sequence is further denoted as U.

(3) Construct the sequences $V = v^R, \mathsf{N}, v$, $W = \mathsf{N}-1, S_{\mathsf{N}-2}, \mathsf{N}$, and let W' be W where the first two elements have been exchanged.

(4) The transition sequence S_{N} is thus the concatenation U^R, V, W'.

It has been proven in [SZ04] that S_{N} is transition sequence of a cyclic N-bits Gray code if $S_{\mathsf{N}-2}$ is. However, the step (1) is not a constructive step that precises how to select the subsequences which ensures that yielded Gray code is balanced. Next section shows how to choose the sequence l to have the balance property.

5.2. Balanced Codes

Let us first recall how to formalize the balance property of a Gray code. Let $L = w_1, w_2, \ldots, w_{2^{\mathsf{N}}}$ be the sequence of a N-bits cyclic Gray code. The transition sequence $S = s_1, s_2, \ldots, s_{2^n}$, s_i, $1 \leq i \leq 2^{\mathsf{N}}$, indicates which bit position changes between codewords at index i and $i+1$ modulo 2^{N}. The *transition count* function $TC_{\mathsf{N}} : \{1, \ldots, \mathsf{N}\} \rightarrow \{0, \ldots, 2^{\mathsf{N}}\}$ gives the number of times i occurs in S, *i.e.*, the number of times the bit i has been switched in L.

The Gray code is *totally balanced* if TC_{N} is constant (and equal to $\frac{2^{\mathsf{N}}}{\mathsf{N}}$). It is *balanced* if for any two bit indices i and j, $|TC_{\mathsf{N}}(i) - TC_{\mathsf{N}}(j)| \leq 2$.

Running Example. *Let $L^* = 000, 100, 101, 001, 011, 111, 110, 010$ be the Gray code that corresponds to the Hamiltonian cycle that has been removed in f^*. Its transition sequence is $S = 3, 1, 3, 2, 3, 1, 3, 2$ and its transition count function is $TC_3(1) = TC_3(2) = 2$ and $TC_3(3) = 4$. Such a Gray code is balanced.*

Let now $L^4 = 0000, 0010, 0110, 1110, 1111, 0111, 0011, 0001, 0101, 0100, 1100, 1101, 1001, 1011, 1010, 1000$ be a cyclic Gray code. Since $S = 2, 3, 4, 1, 4, 3, 2, 3, 1, 4, 1, 3, 2, 1, 2, 4$ TC_4 is equal to 4 everywhere, this code is thus totally balanced.

On the contrary, for the standard 4-bits Gray code $L^{st} = 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000$, we have $TC_4(1) = 8$ $TC_4(2) = 4$ $TC_4(3) = TC_4(4) = 2$ and the code is neither balanced nor totally balanced.

Theorem 5.1. *Let N in $\mathbb{N}^*$, and a_{N} be defined by $a_{\mathsf{N}} = 2\lfloor \frac{2^{\mathsf{N}}}{2\mathsf{N}} \rfloor$. There exists then a sequence l in step (1) of the* Robinson-Cohn extension *algorithm such that all the transition counts $TC_{\mathsf{N}}(i)$ are a_{N} or $a_{\mathsf{N}}+2$ for any i, $1 \leq i \leq \mathsf{N}$.*

The proof is done by induction on N. Let us immediately verify that it is established for both odd and even smallest values, *i.e.* 3 and 4. For the initial case where $\mathsf{N} = 3$, *i.e.* $\mathsf{N}-2 = 1$ we successively have: $S_1 = 1, 1$, $l = 2$, $u_0 = \emptyset$, and $v = \emptyset$. Thus again the algorithm successively produces $U = 1, 2, 1$, $V = 3$, $W = 2, 1, 1, 3$, and $W' = 1, 2, 1, 3$. Finally, S_3 is $1, 2, 1, 3, 1, 2, 1, 3$ which obviously verifies the theorem. For the initial case where $\mathsf{N} = 4$, *i.e.* $\mathsf{N}-2 = 2$ we successively have: $S_1 = 1, 2, 1, 2$, $l = 4$, $u_0, u_1, u_2 = \emptyset, \emptyset, \emptyset$, and $v = \emptyset$. Thus again the algorithm successively produces $U = 1,3,2,3,4,1,4,3,2$, $V = 4$, $W = 3,1,2,1,2,4$, and $W' = 1,3,2,1,2,4$. Finally, S_4 is $2,3,4,1,4,3,2,3,1,4,1,3,2,1,2,4$ such that $TC_4(i) = 4$ and the theorem is established for odd and even initial values.

For the inductive case, let us first define some variables. Let c_N (resp. d_N) be the number of elements whose transition count is exactly a_N (resp $a_\mathsf{N}+2$). These two variables are defined by the system

$$\left\{\begin{array}{lcl} c_\mathsf{N} + d_\mathsf{N} & = & \mathsf{N} \\ c_\mathsf{N} a_\mathsf{N} + d_\mathsf{N}(a_\mathsf{N}+2) & = & 2^\mathsf{N} \end{array}\right. \quad \Leftrightarrow \quad \left\{\begin{array}{lcl} d_\mathsf{N} & = & \dfrac{2^\mathsf{N} - \mathsf{N}.a_\mathsf{N}}{2} \\ c_\mathsf{N} & = & \mathsf{N} - d_\mathsf{N} \end{array}\right.$$

Since a_N is even, d_N is an integer. Let us first proove that both c_N and d_N are positive integers. Let q_N and r_N, respectively, be the quotient and the remainder in the Euclidean disvision of 2^N by $2\mathsf{N}$, *i.e.* $2^\mathsf{N} = q_\mathsf{N}.2\mathsf{N} + r_\mathsf{N}$, with $0 \leq r_\mathsf{N} < 2\mathsf{N}$. First of all, the integer r is even since $r_\mathsf{N} = 2^\mathsf{N} - q_\mathsf{N}.2\mathsf{N} = 2(2^{\mathsf{N}-1} - q_\mathsf{N}.\mathsf{N})$. Next, a_N is $\frac{2^\mathsf{N} - r_\mathsf{N}}{\mathsf{N}}$. Consequently d_N is $r_\mathsf{N}/2$ and is thus a positive integer s.t. $0 \leq d_\mathsf{N} < \mathsf{N}$. The proof for c_N is obvious.

For any i, $1 \leq i \leq \mathsf{N}$, let zi_N (resp. ti_N and bi_N) be the occurence number of element i in the sequence $u_0, \ldots, u_{l-2}$ (resp. in the sequences $s_{i_1}, \ldots, s_{i_l}$ and v) in step (1) of the algorithm.

Due to the definition of u' in step (2), $3.zi_\mathsf{N} + ti_\mathsf{N}$ is the number of element i in the sequence U. It is clear that the number of element i in the sequence V is $2bi_\mathsf{N}$ due to step (3). We thus have the following system:

$$\left\{\begin{array}{lcl} 3.zi_\mathsf{N} + ti_\mathsf{N} + 2.bi_\mathsf{N} + TC_{\mathsf{N}-2}(i) & = & TC_\mathsf{N}(i) \\ zi_\mathsf{N} + ti_\mathsf{N} + bi_\mathsf{N} & = & TC_{\mathsf{N}-2}(i) \end{array}\right. \quad \Leftrightarrow$$

$$\left\{\begin{array}{lcl} zi_\mathsf{N} & = & \dfrac{TC_\mathsf{N}(i) - 2.TC_{\mathsf{N}-2}(i) - bi_\mathsf{N}}{2} \\ ti_\mathsf{N} & = & TC_{\mathsf{N}-2}(i) - zi_\mathsf{N} - bi_\mathsf{N} \end{array}\right. \tag{3}$$

In this set of 2 equations with 3 unknown variables, let b_i be set with 0. In this case, since TC_N is even (equal to a_N or to $a_\mathsf{N}+2$), the variable zi_N is thus an integer. Let us now prove that the resulting system has always positive integer solutions z_i, t_i, $0 \leq z_i, t_i \leq TC_{\mathsf{N}-2}(i)$ and s.t. their sum is equal to $TC_{\mathsf{N}-2}(i)$. This latter consraint is obviously established if the system has a solution. We thus have the following system.

$$\left\{\begin{array}{lcl} zi_\mathsf{N} & = & \dfrac{TC_\mathsf{N}(i) - 2.TC_{\mathsf{N}-2}(i)}{2} \\ ti_\mathsf{N} & = & TC_{\mathsf{N}-2}(i) - zi_\mathsf{N} \end{array}\right. \tag{4}$$

The definition of $TC_{\mathsf{N}}(i)$ depends on the value of N. When $3 \leq N \leq 7$, values are defined as follows:

$$
\begin{array}{rcl}
TC_3 & = & [2,2,4] \\
TC_5 & = & [6,6,8,6,6] \\
TC_7 & = & [18,18,20,18,18,18,18] \\
 & & \\
TC_4 & = & [4,4,4,4] \\
TC_6 & = & [10,10,10,10,12,12]
\end{array}
$$

It is not hard to verify that all these instanciations verify the aformentioned contraints.

When $N \geq 8$, $TC_{\mathsf{N}}(i)$ is defined as follows:

$$
TC_{\mathsf{N}}(i) = \left\{ \begin{array}{l} a_{\mathsf{N}} \text{ if } 1 \leq i \leq c_{\mathsf{N}} \\ a_{\mathsf{N}} + 2 \text{ if } c_{\mathsf{N}} + 1 \leq i \leq c_{\mathsf{N}} + d_{\mathsf{N}} \end{array} \right. \tag{5}
$$

We thus have

$$
\begin{array}{rcl}
TC_{\mathsf{N}}(i) - 2.TC_{\mathsf{N}-2}(i) & \geq & a_{\mathsf{N}} - 2(a_{\mathsf{N}-2} + 2) \\
 & \geq & \frac{2^{\mathsf{N}} - r_{\mathsf{N}}}{\mathsf{N}} - 2\left(\frac{2^{\mathsf{N}-2} - r_{\mathsf{N}-2}}{\mathsf{N}-2} + 2\right) \\
 & \geq & \frac{2^{\mathsf{N}} - 2N}{\mathsf{N}} - 2\left(\frac{2^{\mathsf{N}-2}}{\mathsf{N}-2} + 2\right) \\
 & \geq & \dfrac{(\mathsf{N}-2).2^{\mathsf{N}} - 2N.2^{\mathsf{N}-2} - 6N(N-2)}{\mathsf{N}.(\mathsf{N}-2)}
\end{array}
$$

A simple variation study of the function $t : \mathbb{R} \to \mathbb{R}$ such that $x \mapsto t(x) = (x-2).2^x - 2x.2^{x-2} - 6x(x-2)$ shows that its derivative is strictly postive if $x \geq 6$ and $t(8) = 224$. The integer $TC_{\mathsf{N}}(i) - 2.TC_{\mathsf{N}-2}(i)$ is thus positive for any $\mathsf{N} \geq 8$ and the proof is established.

For each element i, we are then left to choose zi_{N} positions among $TC_{\mathsf{N}}(i)$, which leads to $\binom{TC_{\mathsf{N}}(i)}{zi_{\mathsf{N}}}$ possibilities. Notice that all such choices lead to a hamiltonian path.

6. Stopping Time

This section considers functions $f : \mathbb{B}^{\mathsf{N}} \to \mathbb{B}^{\mathsf{N}}$ issued from an hypercube where an Hamiltonian path has been removed as described in previous section. Notice that the iteration graph is always a subgraph of N-cube augmented with all the self-loop, *i.e.*, all the edges (v, v) for any $v \in \mathbb{B}^{\mathsf{N}}$. Next, if we add probabilities on the transition graph, iterations can be interpreted as Markov chains.

Running Example. *Let us consider for instance the graph* $\Gamma(f)$ *defined in* FIGURE 1. *and the probability function* p *defined on the set of edges as follows:*

$$p(e) \left\{ \begin{array}{l} = \frac{2}{3} \textit{ if } e=(v,v) \textit{ with } v \in \mathbb{B}^3, \\ = \frac{1}{6} \textit{ otherwise.} \end{array} \right.$$

The matrix P *of the Markov chain associated to the function* f^* *and to its probability function* p *is*

$$P = \frac{1}{6} \left(\begin{array}{llllllll} 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 4 \end{array} \right)$$

A specific random walk in this modified hypercube is first introduced (See section 6.1). We further theoretical study this random walk to provide a upper bound of fair sequences (See section 6.2). We finally complete these study with experimental results that reduce this bound (Sec. **??**). Notice that for a general references on Markov chains see [LPW06], and particularly Chapter 5 on stopping times.

6.1. Formalizing the Random Walk

First of all, let π, μ be two distributions on $\mathbb{B}^{\mathsf{N}}$. The total variation distance between π and μ is denoted $\|\pi-\mu\|_{\rm TV}$ and is defined by

$$\|\pi-\mu\|_{\rm TV} = \max_{A\subset \mathbb{B}^{\mathsf{N}}} |\pi(A)-\mu(A)|.$$

It is known that

$$\|\pi-\mu\|_{\rm TV} = \frac{1}{2}\sum_{X\in\mathbb{B}^{\mathsf{N}}} |\pi(X)-\mu(X)|.$$

Moreover, if ν is a distribution on $\mathbb{B}^{\mathsf{N}}$, one has

$$\|\pi-\mu\|_{\rm TV} \leq \|\pi-\nu\|_{\rm TV} + \|\nu-\mu\|_{\rm TV}$$

Let P be the matrix of a Markov chain on $\mathbb{B}^{\mathsf{N}}$. $P(X,\cdot)$ is the distribution induced by the X-th row of P. If the Markov chain induced by P has a stationary distribution π, then we define

$$d(t) = \max_{X\in\mathbb{B}^{\mathsf{N}}} \|P^t(X,\cdot)-\pi\|_{\rm TV}.$$

and

$$t_{\rm mix}(\varepsilon) = \min\{t \mid d(t) \leq \varepsilon\}.$$

Intuitively speaking, $t_{\rm mix}$ is a mixing time *i.e.*, is the time until the matrix X of a Markov chain is ϵ-close to a stationary distribution.

One can prove that

$$t_{\rm mix}(\varepsilon) \leq \lceil \log_2(\varepsilon^{-1}) \rceil t_{\rm mix}(\frac{1}{4})$$

Let $(X_t)_{t\in\mathbb{N}}$ be a sequence of $\mathbb{B}^{\mathsf{N}}$ valued random variables. A $\mathbb{N}$-valued random variable τ is a *stopping time* for the sequence (X_i) if for each t there exists $B_t \subseteq (\mathbb{B}^{\mathsf{N}})^{t+1}$ such that $\{\tau = t\} = \{(X_0, X_1, \ldots, X_t) \in B_t\}$. In other words, the event $\{\tau = t\}$ only depends on the values of $(X_0, X_1, \ldots, X_t)$, not on X_k with $k > t$.

Let $(X_t)_{t\in\mathbb{N}}$ be a Markov chain and $f(X_{t-1}, Z_t)$ a random mapping representation of the Markov chain. A *randomized stopping time* for the Markov chain is a stopping time for $(Z_t)_{t\in\mathbb{N}}$. If the Markov chain is irreducible and has π as stationary distribution, then a *stationary time* τ is a randomized stopping time (possibly depending on the starting position X), such that the distribution of X_τ is π:

$$\mathbb{P}_X(X_\tau = Y) = \pi(Y).$$

6.2. Upper bound of Stopping Time

A stopping time τ is a *s*trong stationary time if X_τ is independent of τ.

Theorem 6.1. *If τ is a strong stationary time, then $d(t) \leq \max_{X\in\mathbb{B}^{\mathsf{N}}} \mathbb{P}_X(\tau > t)$.*

Let $E = \{(X, Y) \mid X \in \mathbb{B}^{\mathsf{N}}, Y \in \mathbb{B}^{\mathsf{N}},\ X = Y \text{ or } X \oplus Y \in 0^*10^*\}$. In other words, E is the set of all the edges in the classical N-cube. Let h be a function from $\mathbb{B}^{\mathsf{N}}$ into $[\![1, \mathsf{N}]\!]$. Intuitively speaking h aims at memorizing for each node $X \in \mathbb{B}^{\mathsf{N}}$ which edge is removed in the Hamiltonian cycle, *i.e.* which bit in $[\![1, \mathsf{N}]\!]$ cannot be switched.

We denote by E_h the set $E \setminus \{(X, Y) \mid X \oplus Y = 0^{\mathsf{N}-h(X)}10^{h(X)-1}\}$. This is the set of the modified hypercube, *i.e.*, the N-cube where the Hamiltonian cycle h has been removed.

We define the Markov matrix P_h for each line X and each column Y as follows:

$$\left\{\begin{array}{ll} P_h(X,X) = \frac{1}{2} + \frac{1}{2\mathsf{N}} & \\ P_h(X,Y) = 0 & \text{if } (X,Y) \notin E_h \\ P_h(X,Y) = \frac{1}{2\mathsf{N}} & \text{if } X \neq Y \text{ and } (X,Y) \in E_h \end{array}\right. \tag{6}$$

We denote by $\overline{h} : \mathbb{B}^{\mathsf{N}} \to \mathbb{B}^{\mathsf{N}}$ the function such that for any $X \in \mathbb{B}^{\mathsf{N}}$, $(X, \overline{h}(X)) \in E$ and $X \oplus \overline{h}(X) = 0^{\mathsf{N}-h(X)}10^{h(X)-1}$. The function $\overline{h}$ is said *square-free* if for every $X \in \mathbb{B}^{\mathsf{N}}$, $\overline{h}(\overline{h}(X)) \neq X$.

Lemma 6.2. *If $\overline{h}$ is bijective and square-free, then $h(\overline{h}^{-1}(X)) \neq h(X)$.*

Proof. Let $\overline{h}$ be bijective. Let $k \in [\![1, \mathsf{N}]\!]$ s.t. $h(\overline{h}^{-1}(X)) = k$. Then $(\overline{h}^{-1}(X), X)$ belongs to E and $\overline{h}^{-1}(X) \oplus X = 0^{\mathsf{N}-k}10^{k-1}$. Let us suppose $h(X) = h(\overline{h}^{-1}(X))$. In such a case, $h(X) = k$. By definition of $\overline{h}$, $(X, \overline{h}(X)) \in E$ and $X \oplus \overline{h}(X) = 0^{\mathsf{N}-h(X)}10^{h(X)-1} = 0^{\mathsf{N}-k}10^{k-1}$. Thus $\overline{h}(X) = \overline{h}^{-1}(X)$, which leads to $\overline{h}(\overline{h}(X)) = X$. This contradicts the square-freeness of $\overline{h}$. □

Let Z be a random variable that is uniformly distributed over $[\![1, \mathsf{N}]\!] \times \mathbb{B}$. For $X \in \mathbb{B}^{\mathsf{N}}$, we define, with $Z = (i, b)$,

$$\left\{ \begin{array}{ll} f(X, Z) = X \oplus (0^{\mathsf{N}-i}10^{i-1}) & \text{if } b = 1 \text{ and } i \neq h(X), \\ f(X, Z) = X & \text{otherwise.} \end{array} \right.$$

The Markov chain is thus defined as

$$X_t = f(X_{t-1}, Z_t)$$

An integer $\ell \in [\![1, \mathsf{N}]\!]$ is said *fair* at time t if there exists $0 \leq j < t$ such that $Z_{j+1} = (\ell, \cdot)$ and $h(X_j) \neq \ell$. In other words, there exist a date j before t where the first element of the random variable Z is exactly l (*i.e.*, l is the strategy at date j) and where the configuration X_j allows to traverse the edge l.

Let τ_{stop} be the first time all the elements of $[\![1, \mathsf{N}]\!]$ are fair. The integer τ_{stop} is a randomized stopping time for the Markov chain (X_t).

Lemma 6.3. *The integer τ_{stop} is a strong stationary time.*

Proof. Let τ_ℓ be the first time that ℓ is fair. The random variable Z_{τ_ℓ} is of the form (ℓ, b) such that $b = 1$ with probability $\frac{1}{2}$ and $b = 0$ with probability $\frac{1}{2}$. Since $h(X_{\tau_\ell - 1}) \neq \ell$ the value of the ℓ-th bit of X_{τ_ℓ} is 0 or 1 with the same probability ($\frac{1}{2}$). This probability is independent of the value of the other bits.

Moving next in the chain, at each step, the l-th bit is switched from 0 to 1 or from 1 to 0 each time with the same probability. Therefore, for $t \geq \tau_\ell$, the ℓ-th bit of X_t is 0 or 1 with the same probability, proving the lemma. □

Theorem 6.4. *If $\overline{h}$ is bijective and square-free, then $E[\tau_{\text{stop}}] \leq 8\mathsf{N}^2 + 4\mathsf{N}\ln(\mathsf{N}+1)$.*

For each $X \in \mathbb{B}^{\mathsf{N}}$ and $\ell \in [\![1, \mathsf{N}]\!]$, let $S_{X,\ell}$ be the random variable that counts the number of steps from X until we reach a configuration where ℓ is fair. More formally

$$S_{X,\ell} = \min\{t \geq 1 \mid h(X_{t-1}) \neq \ell \text{ and } Z_t = (\ell, .) \text{ and } X_0 = X\}.$$

Lemma 6.5. *Let $\overline{h}$ is a square-free bijective function. Then for all X and all ℓ, the inequality $E[S_{X,\ell}] \leq 8\mathsf{N}^2$ is established.*

Proof. For every X, every ℓ, one has $\mathbb{P}(S_{X,\ell}) \leq 2) \geq \frac{1}{4\mathsf{N}^2}$. Let $X_0 = X$. Indeed,

- if $h(X) \neq \ell$, then $\mathbb{P}(S_{X,\ell} = 1) = \frac{1}{2\mathsf{N}} \geq \frac{1}{4\mathsf{N}^2}$.

- otherwise, $h(X) = \ell$, then $\mathbb{P}(S_{X,\ell} = 1) = 0$. But in this case, intuitively, it is possible to move from X to $\overline{h}^{-1}(X)$ (with probability $\frac{1}{2N}$). And in $\overline{h}^{-1}(X)$ the l-th bit can be switched. More formally, since $\overline{h}$ is square-free, $\overline{h}(X) = \overline{h}(\overline{h}(\overline{h}^{-1}(X))) \neq \overline{h}^{-1}(X)$. It follows that $(X, \overline{h}^{-1}(X)) \in E_h$. We thus have $P(X_1 = \overline{h}^{-1}(X)) = \frac{1}{2\mathsf{N}}$. Now, by Lemma 6.2, $h(\overline{h}^{-1}(X)) \neq h(X)$. Therefore $\mathbb{P}(S_{x,\ell} = 2 \mid X_1 = \overline{h}^{-1}(X)) = \frac{1}{2\mathsf{N}}$, proving that $\mathbb{P}(S_{x,\ell} \leq 2) \geq \frac{1}{4\mathsf{N}^2}$.

Therefore, $\mathbb{P}(S_{X,\ell} \geq 3) \leq 1 - \frac{1}{4\mathsf{N}^2}$. By induction, one has, for every i, $\mathbb{P}(S_{X,\ell} \geq 2i) \leq \left(1 - \frac{1}{4\mathsf{N}^2}\right)^i$. Moreover, since $S_{X,\ell}$ is positive, it is known [MU05, lemma 2.9], that

$$E[S_{X,\ell}] = \sum_{i=1}^{+\infty} \mathbb{P}(S_{X,\ell} \geq i).$$

Since $\mathbb{P}(S_{X,\ell} \geq i) \geq \mathbb{P}(S_{X,\ell} \geq i+1)$, one has

$$E[S_{X,\ell}] = \sum_{i=1}^{+\infty} \mathbb{P}(S_{X,\ell} \geq i) \leq \mathbb{P}(S_{X,\ell} \geq 1) + \mathbb{P}(S_{X,\ell} \geq 2) + 2\sum_{i=1}^{+\infty} \mathbb{P}(S_{X,\ell} \geq 2i).$$

Consequently,

$$E[S_{X,\ell}] \leq 1 + 1 + 2\sum_{i=1}^{+\infty} \left(1 - \frac{1}{4\mathsf{N}^2}\right)^i = 2 + 2(4\mathsf{N}^2 - 1) = 8\mathsf{N}^2,$$

which concludes the proof. □

Let τ'_{stop} be the time used to get all the bits but one fair.

Lemma 6.6. *One has* $E[\tau'_{\text{stop}}] \leq 4\mathsf{N}\ln(\mathsf{N}+1)$.

Proof. This is a classical Coupon Collector's like problem. Let W_i be the random variable counting the number of moves done in the Markov chain while we had exactly $i-1$ fair bits. One has $\tau'_{\text{stop}} = \sum_{i=1}^{\mathsf{N}-1} W_i$. But when we are at position X with $i-1$ fair bits, the probability of obtaining a new fair bit is either $1 - \frac{i-1}{\mathsf{N}}$ if $h(X)$ is fair, or $1 - \frac{i-2}{\mathsf{N}}$ if $h(X)$ is not fair.

Therefore, $\mathbb{P}(W_i = k) \leq \left(\frac{i-1}{\mathsf{N}}\right)^{k-1} \frac{\mathsf{N}-i+2}{\mathsf{N}}$. Consequently, we have $\mathbb{P}(W_i \geq k) \leq \left(\frac{i-1}{\mathsf{N}}\right)^{k-1} \frac{\mathsf{N}-i+2}{\mathsf{N}-i+1}$. It follows that $E[W_i] = \sum_{k=1}^{+\infty} \mathbb{P}(W_i \geq k) \leq \mathsf{N}\frac{\mathsf{N}-i+2}{(\mathsf{N}-i+1)^2} \leq \frac{4\mathsf{N}}{\mathsf{N}-i+2}$.

It follows that $E[W_i] \leq \frac{4\mathsf{N}}{\mathsf{N}-i+2}$. Therefore

$$E[\tau'_{\text{stop}}] = \sum_{i=1}^{\mathsf{N}-1} E[W_i] \leq 4\mathsf{N}\sum_{i=1}^{\mathsf{N}-1} \frac{1}{\mathsf{N}-i+2} = 4\mathsf{N}\sum_{i=3}^{\mathsf{N}+1} \frac{1}{i}.$$

But $\sum_{i=1}^{\mathsf{N}+1} \frac{1}{i} \leq 1 + \ln(\mathsf{N}+1)$. It follows that $1 + \frac{1}{2} + \sum_{i=3}^{\mathsf{N}+1} \frac{1}{i} \leq 1 + \ln(\mathsf{N}+1)$. Consequently, $E[\tau'_{\text{stop}}] \leq 4\mathsf{N}(-\frac{1}{2} + \ln(\mathsf{N}+1)) \leq 4\mathsf{N}\ln(\mathsf{N}+1)$. □

One can now prove Theorem 6.4.

Proof. Since τ'_{stop} is the time used to obtain $\mathsf{N}-1$ fair bits. Assume that the last unfair bit is ℓ. One has $\tau_{\text{stop}} = \tau'_{\text{stop}} + S_{X_\tau,\ell}$, and therefore $E[\tau_{\text{stop}}] = E[\tau'_{\text{stop}}] + E[S_{X_\tau,\ell}]$. Therefore, Theorem 6.4 is a direct application of lemma 6.5 and 6.6. □

Notice that the calculus of the stationary time upper bound is obtained under the following constraint: for each vertex in the N-cube there are one ongoing arc and one outgoing arc that are removed. The calculus does not consider (balanced) Hamiltonian cycles, which are more regular and more binding than this constraint. In this later context, we claim that the upper bound for the stopping time should be reduced. This fact is studied in the next section.

6.3. Practical Evaluation of Stopping Times

Let be given a function $f : \mathbb{B}^{\mathsf{N}} \rightarrow \mathbb{B}^{\mathsf{N}}$ and an initial seed x^0. The pseudo code given in algorithm 2 returns the smallest number of iterations such that all elements $\ell \in [\![1, \mathsf{N}]\!]$ are fair. It allows to deduce an approximation of $E[\tau_{\text{stop}}]$ by calling this code many times with many instances of function and many seeds.

Practically speaking, for each number N,$3 \leq \mathsf{N} \leq 16$, 10 functions have been generaed according to method presented in section 5. For each of them, the calculus of the approximation of $E[\tau_{\text{stop}}]$ is executed 10000 times with a random seed. The table 1 summarizes results. It can be observed that the approximation is largely smaller than the upper bound given in theorem 6.4.

Input: a function f, an initial configuration x^0 (N bits)
Output: a number of iterations *nbit*

```
nbit ← 0;
x ← x^0;
visited ← ∅;
while |visited| < N do
    s ← Random(n) ;
    image ← f(x);
    if x[s] ≠ image[s] then
        visited ← visited ∪ {s}
    end
    x[s] ← image[s];
    nbit ← nbit + 1;
end
return nbit;
```

Algorithm 2: Pseudo Code of the stoping time calculus

N	3	4	5	6	7	8	9	10	11	12	13	14	15	16
N	3	10.9	5	17.7	7	25	9	32.7	11	40.8	13	49.2	15	16

TABLE 1. Average Stopping Time

7. EXPERIMENTS

Let us finally present the pseudorandom number generator $\chi_{\textit{15Rairo}}$, which is based on random walks in $\Gamma_{\{b\}}(f)$. More precisely, let be given a Boolean map $f : \mathbb{B}^{\mathsf{N}} \rightarrow \mathbb{B}^{\mathsf{N}}$, a PRNG *Random*, an integer b that corresponds to an iteration number (*i.e.*, the length of the walk), and an initial configuration x^0. Starting from x^0, the algorithm repeats b times a random choice of which edge to follow, and traverses this edge provided it is allowed to do so, *i.e.*, when *Random*(1) is not null. The final configuration is thus outputted. This PRNG is formalized in Algorithm 3.

Input: a function f, an iteration number b, an initial configuration x^0 (n bits)
Output: a configuration x (n bits)

```
x ← x^0;
for i = 0, ..., b − 1 do
    if Random(1) ≠ 0 then
        s ← Random(n);
        x ← F_f(s, x);
    end
end
return x;
```

Algorithm 3: Pseudo Code of the $\chi_{\textit{15Rairo}}$ PRNG

This PRNG is slightly different from $\chi_{\textit{14Secrypt}}$ recalled in Algorithm 1. As this latter, the length of the random walk of our algorithm is always constant (and is equal to b). However, in the current version, we add the constraint that the probability to execute the function F_f is equal to 0.5 since the output of *Random(1)* is uniform in $\{0, 1\}$. This constraint is added to match the theoretical framework of Sect. 6.

Notice that the chaos property of G_f given in Sect.3 only requires that the graph $\Gamma_{\{b\}}(f)$ is strongly connected. Since the $\chi_{\textit{15Rairo}}$ algorithm only adds probability constraints on existing edges, it preserves this property.

For each number $\mathsf{N} = 4, 5, 6, 7, 8$ of bits, we have generated the functions according to the method given in Sect. 4. For each N, we have then restricted this evaluation to the function whose Markov Matrix (issued from Eq. (6)) has the smallest practical mixing time. Such functions are given in Table 2. In this table,

Function f	$f(x)$, for x in $(0,1,2,\ldots,2^n-1)$	N	b	$E[\tau]$
ⓐ	[13,10,9,14,3,11,1,12,15,4,7,5,2,6,0,8]	4	64	154
ⓑ	[29, 22, 25, 30, 19, 27, 24, 16, 21, 6, 5, 28, 23, 26, 1, 17, 31, 12, 15, 8, 10, 14, 13, 9, 3, 2, 7, 20, 11, 18, 0, 4]	5	78	236
ⓒ	[55, 60, 45, 44, 58, 62, 61, 48, 53, 50, 52, 36, 59, 34, 33, 49, 15, 42, 47, 46, 35, 10, 57, 56, 7, 54, 39, 37, 51, 2, 1, 40, 63, 26, 25, 30, 19, 27, 17, 28, 31, 20, 23, 21, 18, 22, 16, 24, 13, 12, 29, 8, 43, 14, 41, 0, 5, 38, 4, 6, 11, 3, 9, 32]	6	88	335
ⓓ	[111, 94, 93, 116, 122, 90, 125, 88, 115, 126, 119, 84, 123, 98, 81, 120, 109, 106, 105, 110, 99, 107, 104, 72, 71, 118, 117, 96, 103, 102, 113, 64, 79, 86, 95, 124, 83, 91, 121, 24, 85, 22, 69, 20, 19, 114, 17, 112, 77, 76, 13, 108, 74, 10, 9, 73, 67, 66, 101, 100, 75, 82, 97, 0, 127, 54, 57, 62, 51, 59, 56, 48, 53, 38, 37, 60, 55, 58, 33, 49, 63, 44, 47, 40, 42, 46, 45, 41, 35, 34, 39, 52, 43, 50, 32, 36, 29, 28, 61, 92, 26, 18, 89, 25, 87, 30, 23, 4, 27, 2, 16, 80, 31, 78, 15, 14, 3, 11, 8, 12, 5, 70, 21, 68, 7, 6, 65, 1]	7	99	450
ⓔ	[223, 190, 249, 254, 187, 251, 233, 232, 183, 230, 247, 180, 227, 178, 240, 248, 237, 236, 253, 172, 203, 170, 201, 168, 229, 166, 165, 244, 163, 242, 241, 192, 215, 220, 205, 216, 218, 222, 221, 208, 213, 210, 212, 214, 219, 211, 217, 209, 239, 202, 207, 140, 139, 234, 193, 204, 135, 196, 199, 132, 194, 130, 225, 200, 159, 62, 185, 252, 59, 250, 169, 56, 191, 246, 245, 52, 243, 50, 176, 48, 173, 238, 189, 44, 235, 42, 137, 184, 231, 38, 37, 228, 35, 226, 177, 224, 151, 156, 141, 152, 154, 158, 157, 144, 149, 146, 148, 150, 155, 147, 153, 145, 175, 206, 143, 136, 11, 142, 129, 8, 7, 198, 197, 4, 195, 2, 161, 160, 255, 124, 109, 108, 122, 126, 125, 112, 117, 114, 116, 100, 123, 98, 97, 113, 79, 106, 111, 110, 99, 74, 121, 120, 71, 118, 103, 101, 115, 66, 65, 104, 127, 90, 89, 94, 83, 91, 81, 92, 95, 84, 87, 85, 82, 86, 80, 88, 77, 76, 93, 72, 107, 78, 105, 64, 69, 102, 68, 70, 75, 67, 73, 96, 55, 58, 45, 188, 51, 186, 61, 40, 119, 182, 181, 53, 179, 54, 33, 49, 15, 174, 47, 60, 171, 46, 57, 32, 167, 6, 36, 164, 43, 162, 1, 0, 63, 26, 25, 30, 19, 27, 17, 28, 31, 20, 23, 21, 18, 22, 16, 24, 13, 10, 29, 14, 3, 138, 41, 12, 39, 134, 133, 5, 131, 34, 9, 128]	8	110	582

TABLE 2. Functions with DSCC Matrix and smallest MT

let us consider for instance the function ⓐ from $\mathbb{B}^4$ to $\mathbb{B}^4$ defined by the following images : $[13, 10, 9, 14, 3, 11, 1, 12, 15, 4, 7, 5, 2, 6, 0, 8]$. In other words, the image of 3 (0011) by ⓐ is 14 (1110): it is obtained as the binary value of the fourth element in the second list (namely 14).

In this table the column that is labeled with b (respectively by $E[\tau]$) gives the practical mixing time where the deviation to the standard distribution is lesser than 10^{-6} (resp. the theoretical upper bound of stopping time as described in Sect. 6).

Let us first discuss about results against the NIST test suite. In our experiments, 100 sequences (s = 100) of 1,000,000 bits are generated and tested. If the value $\mathbb{P}_T$ of any test is smaller than 0.0001, the sequences are considered to be not good enough and the generator is unsuitable. Table 3 shows $\mathbb{P}_T$ of sequences based on discrete chaotic iterations using different schemes. If there are at least two statistical values in a test, this test is marked with an asterisk and the average value is computed to characterize the statistics. We can see in Table 3 that all the rates are greater than 97/100, *i.e.*, all the generators achieve to pass the NIST battery of tests.

Method	ⓐ	ⓑ	ⓒ	ⓓ	ⓔ
Frequency (Monobit)	0.851 (0.98)	0.719 (0.99)	0.699 (0.99)	0.514 (1.0)	0.798 (0.99)
Frequency (Monobit)	0.851 (0.98)	0.719 (0.99)	0.699 (0.99)	0.514 (1.0)	0.798 (0.99)
Frequency within a Block	0.262 (0.98)	0.699 (0.98)	0.867 (0.99)	0.145 (1.0)	0.455 (0.99)
Cumulative Sums (Cusum) *	0.301 (0.98)	0.521 (0.99)	0.688 (0.99)	0.888 (1.0)	0.598 (1.0)
Runs	0.224 (0.97)	0.383 (0.97)	0.108 (0.96)	0.213 (0.99)	0.616 (0.99)
Longest Run of 1s	0.383 (1.0)	0.474 (1.0)	0.983 (0.99)	0.699 (0.98)	0.897 (0.96)
Binary Matrix Rank	0.213 (1.0)	0.867 (0.99)	0.494 (0.98)	0.162 (0.99)	0.924 (0.99)
Disc. Fourier Transf. (Spect.)	0.474 (1.0)	0.739 (0.99)	0.012 (1.0)	0.678 (0.98)	0.437 (0.99)
Unoverlapping Templ. Match.*	0.505 (0.990)	0.521 (0.990)	0.510 (0.989)	0.511 (0.990)	0.499 (0.990)
Overlapping Temp. Match.	0.574 (0.98)	0.304 (0.99)	0.437 (0.97)	0.759 (0.98)	0.275 (0.99)
Maurer's Universal Statistical	0.759 (0.96)	0.699 (0.97)	0.191 (0.98)	0.699 (1.0)	0.798 (0.97)
Approximate Entropy (m=10)	0.759 (0.99)	0.162 (0.99)	0.867 (0.99)	0.534 (1.0)	0.616 (0.99)
Random Excursions *	0.666 (0.994)	0.410 (0.962)	0.287 (0.998)	0.365 (0.994)	0.480 (0.985)
Random Excursions Variant *	0.337 (0.988)	0.519 (0.984)	0.549 (0.994)	0.225 (0.995)	0.533 (0.993)
Serial* (m=10)	0.630 (0.99)	0.529 (0.99)	0.460 (0.99)	0.302 (0.995)	0.360 (0.985)
Linear Complexity	0.719 (1.0)	0.739 (0.99)	0.759 (0.98)	0.122 (0.97)	0.514 (0.99)

TABLE 3. NIST SP 800-22 test results ($\mathbb{P}_T$)

8. Conclusion

This work has assumed a Boolean map f which is embedded into a discrete-time dynamical system G_f. This one is supposed to be iterated a fixed number p_1 or p_2,..., or p of times before its output is considered. This work has first shown that iterations of G_f are chaotic if and only if its iteration graph $\Gamma_{\mathcal{P}}(f)$ is strongly connected where $\mathcal{P}$ is $\{p_1, \ldots, p\}$. Any PRNG, which iterates G_f as above satisfies in some cases the property of chaos.

We then have shown that a previously presented approach can be directly applied here to generate function f with strongly connected $\Gamma_{\mathcal{P}}(f)$. The iterated map inside the generator is built by first removing from a N-cube an Hamiltonian path and next adding a self loop to each vertex. The PRNG can thus be seen as a random walks of length in P into N this new cube. We furthermore have exhibit a bound on the number of iterations that are sufficient to obtain a uniform distribution of the output. Finally, experiments through the NIST battery have shown that the statistical properties are almost established for $\mathsf{N} = 4, 5, 6, 7, 8$.

In future work, we intend to understand the link between statistical tests and the properties of chaos for the associated iterations. By doing so, relations between desired statistically unbiased behaviors and topological properties will be understood, leading to better choices in iteration functions. Conditions allowing the reduction of the stopping-time will be investigated too, while other modifications of the hypercube will be regarded in order to enlarge the set of known chaotic and random iterations.

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Communicated by (The editor will be set by the publisher).
(The dates will be set by the publisher).