1 Mathematical Backgroung

Let π, μ be two distribution on a same set Ω. The total variation distance between π and μ is denoted $\|\pi-\mu\|_{\text{TV}}$ and is defined by

$$\|\pi-\mu\|_{\text{TV}} = \max_{A\subset\Omega} |\pi(A)-\mu(A)|.$$

It is known that

$$\|\pi-\mu\|_{\text{TV}} = \frac{1}{2}\sum_{x\in\Omega}|\pi(x)-\mu(x)|.$$

Moreover, if ν is a distribution on Ω, one has

$$\|\pi-\mu\|_{\text{TV}} \leq \|\pi-\nu\|_{\text{TV}} + \|\nu-\mu\|_{\text{TV}}$$

Let P be the matrix of a markov chain on Ω. $P(x,\cdot)$ is the distribution induced by the x-th row of P. If the markov chain induced by P has a stationary distribution π, then we define

$$d(t) = \max_{x\in\Omega}\|P^t(x,\cdot)-\pi\|_{\text{TV}},$$

and

$$t_{\text{mix}}(\varepsilon) = \min\{t \mid d(t)\leq\varepsilon\}.$$

One can prove that

$$t_{\text{mix}}(\varepsilon) \leq \lceil\log_2(\varepsilon^{-1})\rceil t_{\text{mix}}(\frac{1}{4})$$

Let $(X_t)_{t\in\mathbb{N}}$ be a sequence of Ω valued random variables. A $\mathbb{N}$-valued random variable τ is a *stopping time* for the sequence (X_i) if for each t there exists $B_t \subseteq \omega^{t+1}$ such that $\{tau = t\} = \{(X_0, X_1, \ldots, X_t) \in B_t\}$.

Let $(X_t)_{t\in\mathbb{N}}$ be a markov chain and $f(X_{t-1}, Z_t)$ a random mapping representation of the markov chain. A *randomized stopping time* for the markov chain is a stopping time for $(Z_t)_{t\in\mathbb{N}}$. It he markov chain is irreductible and has π as stationary distribution, then a *stationay time* τ is a randomized stopping time (possibily depending on the starting position x), such that the distribution of X_τ is π:

$$\mathbb{P}_x(X_\tau = y) = \pi(y).$$

Proposition 1 *If τ is a strong stationary time, then $d(t) \leq \max_{x\in\Omega}\mathbb{P}_x(\tau > t)$.*

2 Random walk on the modified Hypercube

Let $\Omega = \{0,1\}^N$ be the set of words of length N. Let $E = \{(x,y) \mid x\in\Omega, y\in\Omega,\ x = y \text{ or } x\oplus y \in 0^*10^*\}$. Let h be a function from Ω into $\{1,\ldots,N\}$.

We denote by E_h the set $E\setminus\{(x,y) \mid x\oplus y = 0^{N-h(x)}10^{h(x)-1}\}$. We define the matrix P_h has follows:

$$\left\{\begin{array}{ll} P_h(x,y) = 0 & \text{if } (x,y)\notin E_h \\ P_h(x,x) = \frac{1}{2}+\frac{1}{2N} & \\ P_h(x,x) = \frac{1}{2N} & \text{otherwise} \end{array}\right.$$

We denote by $\overline{h}$ the function from Ω into ω defined by $x\oplus\overline{h}(x) = 0^{N-h(x)}10^{h(x)-1}$. The function $\overline{h}$ is said *square-free* if for every $x\in E$, $\overline{h}(\overline{h}(x)) \neq x$.

Lemma 2 *If $\overline{h}$ is bijective and square-free, then $h(\overline{h}^{-1}(x)) \neq h(x)$.*

PROOF.

□

Let Z be a random variable over $\{1,\ldots,N\}\times\{0,1\}$ uniformaly distributed. For $X\in\Omega$, we define, with $Z=(i,x)$,

$$\begin{cases} f(X,Z)=X\oplus(0^{N-i}10^{i-1}) & \text{if } x=1 \text{ and } i\neq h(X),\\ f(X,Z)=X & \text{otherwise.}\end{cases}$$

3 Stopping time

An integer $\ell\in\{1,\ldots,N\}$ is said *fair* at time t if there exists $0\leq j<t$ such that $Z_j=(\ell,\cdot)$ and $h(X_j)\neq\ell$.

Let τ_{stop} be the first time all the elements of $\{1,\ldots,N\}$ are fair. The integer τ_{stop} is a randomized stopping time for the markov chain (X_t).

Lemma 3 *The integer τ_{stop} is a strong stationnary time.*

PROOF. Let τ_ℓ be the first time that ℓ is fair. The random variable $Z_{\tau_\ell-1}$ is of the form (ℓ,δ) with $\delta\in\{0,1\}$ and $\delta=1$ with probability $\frac{1}{2}$ and $\delta=0$ with probability $\frac{1}{2}$. Since $h(X_{\tau_\ell-1})\neq\ell$ the value of the ℓ-th bit of X_{τ_ℓ} is δ. Moving next in the chain, at each step, the l-th bit is switch from 0 to 1 or from 1 to 0 each time with the same probability. Therefore, for $t\geq\tau_\ell$, the ℓ-th bit of X_t is 0 or 1 with the same probability, proving the lemma. □

Proposition 4 *If $\overline{h}$ is bijective and square-free, then $E[\tau_{\text{stop}}]\leq 8N^2+N\ln(N+1)$.*

For each $x\in\Omega$ and $\ell\in\{1,\ldots,N\}$, let $S_{x,\ell}$ be the random variable counting the number of steps done until reaching from x a state where ℓ is fair. More formaly

$$S_{x,\ell}=\min\{m\geq 1\mid h(X_m)\neq\ell \text{ and } Z_m=\ell \text{ and } X_0=x\}.$$

We denote by

$$\lambda_h=\max_{x,\ell} S_{x,\ell}.$$

Lemma 5 *If $\overline{h}$ is a square-free bijective function, then one has $E[\lambda_h]\leq 8N^2$.*

PROOF. For every X, every ℓ, one has $\mathbb{P}(S_{X,\ell}\leq 2)\geq\frac{1}{4N^2}$. Let $X_0=X$. Indeed, if $h(X)\neq\ell$, then $\mathbb{P}(S_{X,\ell}=1)=\frac{1}{2N}\geq\frac{1}{4N^2}$. If $h(X)=\ell$, then $\mathbb{P}(S_{X,\ell}=1)=0$. But in this case, intutively, it is possible to move from X to $\overline{h}^{-1}(X)$ (with probability $\frac{1}{2N}$). And in $\overline{h}^{-1}(X)$ the l-th bit is switchable. More fromaly, since $\overline{h}$ is square-free, $\overline{h}(x)=\overline{h}(\overline{h}(\overline{h}^{-1}(X)))\neq\overline{h}^{-1}(X)$. It follows that $(X,\overline{h}^{-1}(X))\in E_h$. Thefore $P(X_1=\overline{h}^{-1}(X))=\frac{1}{2N}$. Now, by Lemma 2, $h(\overline{h}^{-1}(X))\neq h(X)$. Therefore $\mathbb{P}(S_{x,\ell}=2\mid X_1=\overline{h}^{-1}(X))=\frac{1}{2N}$, proving that $\mathbb{P}(S_{x,\ell}\leq 2)\geq\frac{1}{4N^2}$.

Therefore, $\mathbb{P}(S_{x,\ell}\geq 3)\leq 1-\frac{1}{4N^2}$. By induction, one has, for every i, $\mathbb{P}(S_{x,\ell}\geq 2i+1)\leq\left(1-\frac{1}{4N^2}\right)^i$. Moreover, since $S_{X,\ell}$ is positive, it is known [?, lemma 2.9], that

$$E[S_{X,\ell}]=\sum_{i=1}^{+\infty}\mathbb{P}(S_{X,\ell}\geq i).$$

Since $\mathbb{P}(S_{X,\ell}\geq i)\geq\mathbb{P}(S_{X,\ell}\geq i+1)$, one has

$$E[S_{X,\ell}]=\sum_{i=1}^{+\infty}\mathbb{P}(S_{X,\ell}\geq i)\leq\mathbb{P}(S_{X,\ell}\geq 1)+\mathbb{P}(S_{X,\ell}\geq 2)+2\sum_{i=1}^{+\infty}\mathbb{P}(S_{x,\ell}\geq 2i).$$

Consequently,

$$E[S_{x,\ell}] \leq 1 + 1 + 2\sum_{i=1}^{+\infty}\left(1 - \frac{1}{4N^2}\right)^i = 2 + 2(4N^2 - 1) = 8N^2,$$

which concludes the proof. □

Let τ'_{stop} be the first time that there are exactly $N-1$ fair elements.

Lemma 6 *One has* $E[\tau'_{\text{stop}}] \leq N\ln(N+1)$.

PROOF. This is a classical Coupon Collector's like problem. Let W_i be the random variable counting the number of moves done in the markov chain while we had exactly $i-1$ fair bits. One has $\tau'_{\text{stop}} = \sum_{i=1}^{N-1} W_i$. But when we are at position x with $i-1$ fair bits, the probability of obtaining a new fair bit is either $1 - \frac{i-1}{N}$ if $h(x)$ is fair, or $1 - \frac{i-2}{N}$ if $h(x)$ is not fair. It follows that $E[W_i] \leq \frac{N}{N-i+2}$. Therefore

$$E[\tau'_{\text{stop}}] = \sum_{i=1}^{N-1} E[W_i] \leq N\sum_{i=1}^{N-1}\frac{1}{N-i+2} = N\sum_{i=3}^{N+1}\frac{1}{i}.$$

But $\sum_{i=1}^{N+1}\frac{1}{i} \leq 1 + \ln(N+1)$. It follows that $1 + \frac{1}{2} + \sum_{i=3}^{N+1}\frac{1}{i} \leq 1 + \ln(N+1)$. Consequently, $E[\tau'_{\text{stop}}] \leq N(-\frac{1}{2} + \ln(N+1)) \leq N\ln(N+1)$. □

One can now prove Proposition 4.

PROOF. One has $\tau_{\text{stop}} \leq \tau'_{\text{stop}} + \lambda_h$. Therefore, Proposition 4 is a direct application of lemma 5 and 6. □