Tight Frame Various Lengths Filters

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ABSTRACT

Dyadic tight frame wavelets have been considered as an alternative to orthogonal wavelets. They allow symmetry, smooth scaling - wavelet functions, and closely approximate shift invariance. In this paper, we consider tight frame symmetric filterbanks with filters of various lengths. The filterbanks are designed using Gröbner basis methods. An applications example is considered for the case of image denoising.

Keywords: wavelets, tight frame, filterbank, Gröbner basis, smoothness, denoising.

1. INTRODUCTION

As is well known, wavelets based on two channels FIR filterbanks and dilation by 2 cannot be both symmetric and orthogonal, except for the trivial case of Haar wavelets. Symmetry is desirable for example in the applications of image processing. By increasing the number of filters in an orthogonal filterbank, and thus increasing the available degrees of freedom, one obtains symmetry. However, the resulting scaling function and wavelets lack smoothness in general, an important feature for good filterbanks. This suggests the use of *tight frame* filterbanks. Such a setup allows for filters associated with short as well as smooth scaling - wavelet functions. In addition, TF filterbanks allow for a dense time-frequency plane and thus are well suited for the applications of noise removal. The theory of tight frame is by now well documented, see for example [4, 8, 9, 15]. Tight frame filterbanks (oversampled filterbanks) have seen use in noise removal applications, see for example [3, 7, 14, 17].

The paper discusses the design of tight frame symmetric even lengths filterbanks consisting of four various-lengths filters $\{h_0, h_1, h_2, h_3\}$ with dilation factor 2. Such filters have been previously discussed in [20] for the orthogonal case but not for the tight frame case. Using Gröbner basis [2, 6], we build various-lengths tight frame symmetric filterbanks. The Gröbner bases are found using Singular [12]. The paper is to briefly discuss tight frame theory, followed by two examples, including an example of image processing noise removal. A comparison with biorthogonal as well as 4-band tight frame symmetric filters is to be discussed.

2. BACKGROUND THEORY

The theory of filterbanks and frames has been discussed and analyzed; see for example [4, 8, 9]. Here we introduce the basic concepts of frame theory. We consider the case where we have a wavelet system with dilation 2 and 3 wavelets defining the following spaces:

$$\begin{aligned}\mathcal{V}_j &= \operatorname*{Span}_n\{\phi(2^j t - n)\},\\ \mathcal{W}_{i,j} &= \operatorname*{Span}_n\{\psi_i(2^j t - n)\},\ i = 1\ldots 3\end{aligned}$$

with

$$\mathcal{V}_j = \mathcal{V}_{j-1} \cup \mathcal{W}_{1,j-1} \cup \mathcal{W}_{2,j-1} \cup \mathcal{W}_{3,j-1}.$$

The corresponding scaling function and wavelets satisfy the following multiresolution equations:

$$\begin{aligned}\phi(t) &= 2\sum_n h_0(n)\phi(2t-n),\\ \psi_i(t) &= 2\sum_n h_i(n)\phi(2t-n),\ i = 1\ldots 3.\end{aligned}$$

We say the filters $\{h_0, h_1, h_2, h_3\}$ define a perfect reconstruction tight frame when the following equations are satisfied:

$$\sum_{n=0}^{3} H_n(z)H_n(z^{-1}) = 1,$$
$$\sum_{n=0}^{3} H_n(-z)H_n(z^{-1}) = 0.$$

In general, the filters are given as follows:

$$\begin{aligned}H_0(z) &= \left(1+z^{-1}\right)^{K_0} Q_0(z)\\ H_l(z) &= \left(1-z^{-1}\right)^{K_i} Q_i(z),\ i = 1\ldots 3.\end{aligned}$$

2.1. Length of Lowpass Filter h_0

From [16] we know that the minimum length of the lowpass filter in a tight frame is $\text{length}\, h_0 \geq K_0 + \min\{K_1, K_2, K_3\}$. The case of dyadic tight frame symmetry results in [1]:

$$\text{length}\, h_0 \geq K_0 + 2\min\{K_1, K_2, K_3\} - 1.$$

2.2. Smoothness vs. K_0

One of the advantages of tight frame filterbanks is the possibility of achieving high K_0 with an accompanying high degree of smoothness ν_2 without a necessarily large support of h_0, as will be shortly seen. It is shown in [18] that the highest possible derivative, ν_2, for a scaling function $\phi(\cdot)$ given the corresponding $h_0(n)$ is bounded by $\nu_2 < K_0$. Smoothness is measured using the Sobolev exponent of a scaling function ϕ, defined as [11, 21]:

$$\nu_2(\phi) := \sup\{\nu_2 : \int_{-\infty}^{\infty} |\Phi(\omega)|^2 (1 + |\omega|^2)^{\nu_2} \mathrm{d}\omega < \infty\}.$$

The actual computation of ν_2 is found using [13], and for the normalization $\sum_n h_0(n) = 1$ we have

$$\nu_2 = -\frac{1}{2} - \frac{1}{2}\log_2 \lambda_{\max}$$

where $\lambda_{\max}$ is the largest eigenvalue of a matrix generated by $(c_{2i-j})_{-N \leq i,j \leq N}$ with $c(z) = Q_0(z)Q_0(z^{-1})$ and $Q_0(z)$ known from $H_0(z) = (1 + z^{-1})^{K_0} Q_0(z)$.

3. EXAMPLE I

We consider the case of the tight frame symmetric filters $\{h_0, h_1, h_2, h_3\}$ with the moments $\{K_0, K_1, K_2, K_3\} = \{5, 3, 2, 3\}$. In addition, we have $\text{length}\, h_0 = \text{length}\, h_1 = 8$, and $\text{length}\, h_2 = \text{length}\, h_3 = 6$. The filters coeffi-

n	$h_0(n)$	$h_1(n)$
0	$-5/2^8$	$-5/2^8$
1	$-7/2^8$	$-7/2^8$
2	$35/2^8$	$35/2^8 \cdot 11$
3	$105/2^8$	$665/2^8 \cdot 11$
4	$105/2^8$	$-665/2^8 \cdot 11$
5	$35/2^8$	$-35/2^8 \cdot 11$
6	$-7/2^8$	$7/2^8$
7	$-5/2^8$	$5/2^8$
n	$h_2(n)$	$h_3(n)$
0	$-5\sqrt{35}/2^5 \cdot 11$	$-25\sqrt{7}/2^7 \cdot 11$
1	$-7\sqrt{35}/2^5 \cdot 11$	$-35\sqrt{7}/2^7 \cdot 11$
2	$3\sqrt{35}/2^5 \cdot 11$	$115\sqrt{7}/2^7 \cdot 11$
3	$3\sqrt{35}/2^5 \cdot 11$	$-115\sqrt{7}/2^7 \cdot 11$
4	$-7\sqrt{35}/2^5 \cdot 11$	$35\sqrt{7}/2^7 \cdot 11$
5	$-5\sqrt{35}/2^5 \cdot 11$	$25\sqrt{7}/2^7 \cdot 11$

Table 1: Coefficients with $K_0 = 5$, $K_2 = 2$, $K_1 = K_3 = 3$.

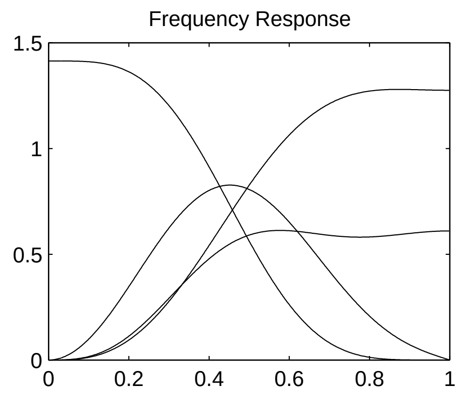

Figure 1: Filters corresponding to $\{K_0, K_1, K_2, K_3\} = \{5, 3, 2, 3\}$

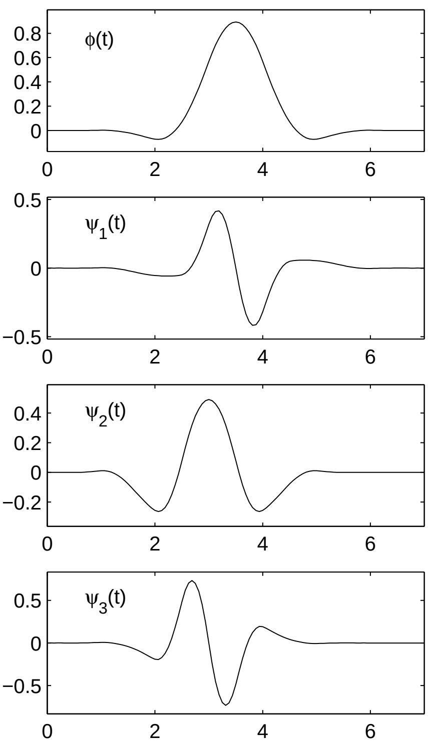

Figure 2: Scaling and wavelet functions corresponding to $\{K_0, K_1, K_2, K_3\} = \{5, 3, 2, 3\}$

cients are listed in table 1. It is interesting to note that the coefficients of h_0 and h_1 are rational and those of h_2 and h_3 are rational within a constant, namely $\sqrt{35}$ in the case of h_2 and $\sqrt{7}$ for the case of h_3. In figure 2 we show the resulting scaling and wavelet functions. Notice how the centers of the wavelets corresponding to ψ_2 and ψ_3 are shifted by $1/2$ relative to the scaling function ϕ and the wavelet ψ_1. The scaling function smoothness coefficient is given by $\nu_2 \approx 3.2596$.

4. EXAMPLE II

We consider in this example the case of the symmetric filterbank with $\{K_0, K_1, K_2, K_3\} = \{7, 3, 2, 3\}$. From the resulting 80th degree Gröbner basis we obtain six reduced and distinct Gröbner bases. We thus obtain a filterbank with $\operatorname{length} h_0 = \operatorname{length} h_1 = 12$, and $\operatorname{length} h_2 = \operatorname{length} h_3 = 8$. The corresponding coefficients are listed in table 2. The scaling function ϕ has smoothness given by $\nu_2 \approx 4.8794$.

As an application example, we consider noise removal from an image. The noise is assumed to be white, Guassian distributed. We use soft threshold approach to remove the noise. Soft thresholding is defined as follows:

$$\hat{x} = \operatorname{sgn}(x)\,(|x| - \eta)_+$$

where η is the estimate of the noise, given by $\eta = \sqrt{2}\sigma_n$ and where σ_n is the standard deviation of the first stage outputs due to $h_1(n_1)h_3(n_2)$, $h_2(n_1)h_3(n_2)$ and $h_3(n_1)h_3(n_2)$. The measure of performance used in this paper is peak signal to noise ratio, given by $PSNR = 10\log_{10}\left(\frac{255^2}{MSE}\right)$, with

$$MSE = \frac{1}{M^2}\sum_{m,j} |x(m,j) - \widehat{x}(m,j)|^2,$$

and where x is an $M \times M$ image and $\widehat{x}$ is the output. For the case of $\sigma_n = 0.05$ we have an image improvement reflected by $PSNR = 79.87$, up from $PSNR = 74.19$ for the noisy image. Fig. 5 depicts the noisy image (top) and the denoised image (below). Similarly, we consider the case where we have noise given by $\sigma_n = 0.075$, and where the noisy image now has $PSNR = 70.66$. Then the resulting image has $PSNR = 77.75$. Table 3 shows a denoising comparison with published symmetric filterbanks.

5. CONCLUSION

In this paper we have introduced a family of filterbanks with various lengths filters designed using Gröbner basis method. Clearly, tight frame filterbanks offer more design degrees of freedom than the orthogonal counterpart. This in turn allows for additional properties such as smoothness and symmetry. Example II shows the performance of such

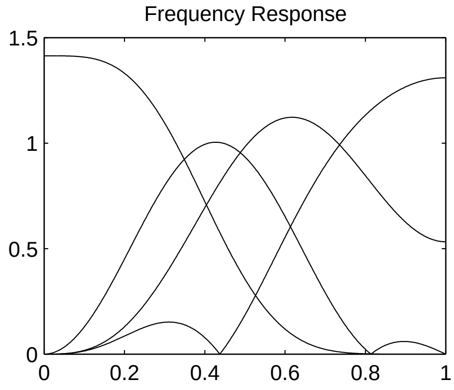

Figure 3: Filters corresponding to $\{K_0, K_1, K_2, K_3\} = \{7, 3, 2, 3\}$

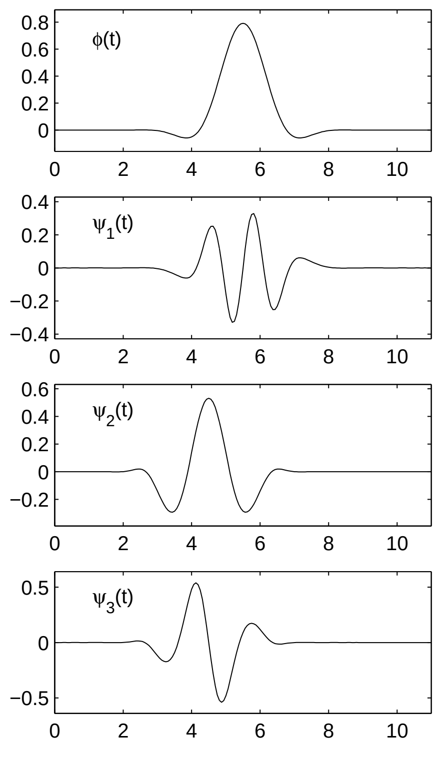

Figure 4: Scaling and wavelet functions corresponding to $\{K_0, K_1, K_2, K_3\} = \{7, 3, 2, 3\}$

Figure 5: Noisy image with $\sigma_n = 0.050$ (top) and the same image with noise removed using soft threshold approach coupled with the filterbank depicted in table 2

n	$h_0(n)$	$h_1(n)$
0	0.000187362	0.000187362
1	−0.006273849	−0.006273849
2	−0.026554550	−0.026554550
3	−0.002060988	−0.002060988
4	0.162938324	0.182947735
5	0.371763701	−0.298252754
6	0.371763701	0.298252754
7	0.162938324	−0.182947735
8	−0.002060988	0.002060988
9	−0.026554550	0.026554550
10	−0.006273849	0.006273849
11	0.000187362	−0.000187362
n	$h_2(n)$	$h_3(n)$
0	0.004103571	0.003056349
1	−0.137408374	−0.102342055
2	−0.096237751	0.049142155
3	0.229542553	0.342889367
4	0.229542553	−0.342889367
5	−0.096237751	−0.049142155
6	−0.137408374	0.102342055
7	0.004103571	−0.003056349

Table 2: Coefficients with $K_0 = 7$, $K_2 = 2$, $K_1 = K_3 = 3$.

Figure 6: Noisy image with $\sigma_n = 0.075$ (top) and the same image with noise removed using soft threshold approach coupled with the filterbank depicted in table 2

Table 3: $PSNR$ in dB resulting from soft thresholding of Lena image with noise variance σ_n^2 using various filterbanks. The first TF column reflects the performance of the filterbank of example II.

σ_n	TF	TF [a]	7/9 [b]	9/15 [c]	6/10 [d]	10/18 [e]
0.050	79.87	79.68	79.50	79.59	79.66	79.44
0.075	77.75	77.60	77.43	77.47	77.45	77.29
0.100	76.19	76.07	75.68	75.87	75.85	75.78

[a] [5]
[b] [9, table 8.3]
[c] [9, table 8.5]
[d] [19, table 2]
[e] [10, table 3]

a filterbank in noise removal applications when compared with published symmetric filters. It is clear that the noise removal approach used in this paper is basic, and better approaches exist. However, the purpose of the example is to depict the performance of the filterbanks discussed in the paper with such applications as noise removal. It would be interesting to couple the filterbanks discussed in this paper with more sophisticated noise removal approaches.

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