Correspondence

An Improved Recursive Median Filtering Scheme for Image Processing

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Abstract—**In a recent publication, it was shown that median filtering is an optimization process in which a two-term cost function is minimized. Based on this functional optimization property of the median filtering process, a new approach for designing the recursive median filter for image processing applications is introduced in this paper. We prove that the new approach is guaranteed to converge to root within a finite number of iterations. The new method is applied to process a real image corrupted by pseudorandom impulsive noise, and the results show that the new scheme provides improved mean square error (MSE) performance over the standard recursive median filters.**

I. Introduction

Median filtering is a nonlinear filtering technique that is known for preserving sharp changes in signal and for being particularly effective in removing impulsive noise. One effective use of median filters has been the reduction of high-frequency and impulsive noise in digital images without the extensive blurring and edge destruction associated with linear filters. Because the median filter is nonlinear, spectral analysis gives little insight into the filtering process. Deterministic and statistical properties of median filters are therefore used to describe the filter's effect on noisy signals. In a recent publication, the author has studied the functional optimization properties of median-related filters. By associating the nonlinear operation of median filtering with a linear cost function, it is shown that median filtering is an optimization process that minimizes a two-term cost function, where one component measures the smoothness between the filter output and its neighborhood points within the filtering window and the other measures the discrepancy between the filter output and the original signal at that particular point. It has been shown that this functional optimization property of the median filtering process can be used to explain why median-related filters have the essential properties of smoothing without extensive blurring of the signal [1].

The 1-D median filter is realized by passing a window over the data, ranking the values in the window, and taking the median as output. Consider a real, discrete-time sequence $\{a(n)\}$, where a is a M-level signal. The output of the median filter $y(n)$ is given by $y(n) = \text{median}[a(n-N), \cdots, a(n), \cdots, a(n+N)]$, where the window contains $2N+1$ points. This is known as a nonrecursive median filter. If we first replace the point $a(n)$ with the output of the median filter before shifting the window to the next position, we have the recursive median filter. The output of the recursive filter is given by $y(n) = \text{median}[y(n-N), \cdots, y(n-1), a(n), \cdots, a(n+N)]$. Recursive median filters have stronger noise attenuating capability than their nonrecursive counterparts and a faster convergence of signals to roots. In fact, any 1-D signal will be reduced to a root after one pass of a recursive median filter [2].

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There are a number of ways to extend the median operation to two dimensions. One way is to pass a 2-D window, such as a square or cross-shaped window, over the 2-D signal. As with the 1-D median filter, the points within the window are ranked and the median values taken as the output. Other methods such as the separable median filter scheme proposed in [3] can also be used.

Two-dimensional median filters have been used with some success in image processing applications. Although noise suppression is obtained, too much signal distortion is introduced, and many features such as thin lines and sharp corners are lost. To overcome these problems, researchers have recently developed several variations of median filters, such as max/median [4] and multistage median [5] filters. In this paper, we introduce a new recursive median filtering scheme for image processing application. This scheme is inspired by the functional optimization properties of median-related filters introduced by the author in [1]. We prove that the new scheme is guaranteed to converge within a finite number of passes and show that it provides improved MSE performance over the standard recursive median filtering schemes.

II. Functional Optimization Property of the Recursive Median Filter

It has been shown in [6] that if a discrete M-level signal $\{a(n)\}$ is thresholded by

$$a_i(n) = \begin{cases} 1 & \text{if } a(n) \geq i \\ 0 & \text{if } a(n) < i \end{cases} \quad \text{for } \leq i \leq M \tag{1}$$

to produce $M-1$ binary sequences, then the output of the recursive median filter $y(n) = \text{median}[y(n-N), \cdots, y(n-1), a(n), \cdots, a(n+N)]$, is equivalent to

$$y(n) = \sum_{i=1}^{M-1} y_i(n) \tag{2}$$

where

$$y_i(n) = \text{median}[y_i(n-N), \cdots, y_i(n-1), a_i(n), \cdots, a_i(n-N)].$$

To avoid the difficulty of using confusing notations and for convenience of analysis, in the rest of this paper we assume that the recursive median filter is sequentially applied to each point of the signal, and we first replace each point with the output of the recursive median filter on that particular point before shifting the window to the next position. For example, if we write $y(n) = \text{median}[a(n-N), \cdots, a(n), \cdots, a(n+N)]$, it is understood that $a(n-N), \cdots, a(n-1)$ are the outputs of the recursive filter applied to those points.

Recursive median filtering of an arbitrary level signal is equivalent to decomposing the signal into binary signals, filtering each binary signal with a binary recursive median filter, and then reversing the decomposition. Therefore, we can show the functional optimization property of the recursive median filtering on multilevel signal by describing such property of the filter on the binary signal.

As in [1], we transfer the $\{0, 1\}$ binary of $\{a_i(n)\}$ into $\{-1, 1\}$ binary of $\{b_i(n)\}$ by the following operations: $b_i(n) = 2a_i(n) - 1$. Define $V_i(n)$ as the output of recursive median filter of $\{b_i(n)\}$, which is given by $V_i(n) = \text{median}[b_i(n-N), \cdots, b_i(n), \cdots, b_i(n+$

$N)]$ and $V_i(n) = 2y_i(n) - 1$. For recursive median filtering of binary sequence $\{b_i(n)\}$, we can write the output of the filter as follows:

$$V_i(n) = \begin{cases} +1 & \text{if } S(n) \geq 0 \\ -1 & \text{otherwise} \end{cases} \quad \text{where } S(n) = \sum_{\substack{j=-N \\ j \neq 0}}^{j=N} b_i(n+j) + b_i(n). \tag{3}$$

It is straightforward to show that the filtering operation of (3) always forces the following function into its minimum:

$$E_i(V_i(n)) = -\sum_{\substack{j=-N \\ j \neq 0}}^{j=N} V_i(n) b_i(n+j) - V_i(n) b_i(n). \tag{4}$$

To confirm the above statement, we observe the following:

$$E_i(+1) = -S(n) \quad \text{and} \quad E_i(-1) = S(n)$$
$$\text{If} \quad V_i(n) = +1, \quad S(n) \geq 0, \quad E_i(+1) \leq E_i(-1)$$
$$\text{If} \quad V_i(n) = -1, \quad S(n) < 0, \quad E_i(-1) < E_i(+1).$$

We call this property of the recursive median filtering the functional optimization property. In each level of the thresholded space, this functional optimization property holds. Thus, we can state that recursive median filtering is an optimization operation in which the output of the filter is always set to the minimum of a cost function of the output state of the filter.

The first term of (4) measures the smoothness between the filter output and its neighborhood points within the filtering window, and the second term measures the discrepancy between the filter output and the original signal. So it is clear that the function of recursive median filtering is a combination of two aspects: The recursive median filtering favors the filtered signal to be smooth and encourages the filtered signal to be the same as the original (noisy) signal.

III. A New Recursive Median Filtering Scheme

Although any 1-D signal will be reduced to a root after one pass of a recursive median filter, this is not the case for 2-D signal such as image [2]. In image processing applications, it is necessary to apply the recursive median filter iteratively.

The process of repeated applications of recursive median filtering can be expressed as follows:

$$V_i^{(t)}(n) = \text{median}\left[V_i^{(t-1)}(n-N), \cdots, V_i^{(t-1)}(n), \cdots, V_i^{(t-1)}(n+N)\right] \tag{5}$$

where the superscript t is the iteration index and $V_i^{(0)}(n) = b_i(n)$.

This process can also be described by the following pseudo-C code. Here, we assume that the total number of signal points is L and at both ends of the signal, N points are appended to allow the filter to reach the edges of the signal.

Algorithm 1

```
Recursive-Median-Filter () {
  for (n = 1; n ≤ L; n++){V_i(n) = b_i(n);}
  do { success = 0;
      for (n = 1; n ≤ L; n++){
      m = median ((V_i(n − N), ···, V_i(n), ···, V_i(n + N))
      if (m == V_i(n)) success++
      V_i(n) = m; }}
  while (success ≠ L); }
```

That is, the output of the tth pass of the filter is obtained by filtering the result of the $(t-1)$th pass. From the above section, we have already seen that the function of recursive median filtering is composed of two elements: First, it smoothes the signal; second, it encourages the filtered signal to resemble the signal to be filtered. So during the process (repeated recursive median filtering), $V_i(n)$ will be smooth and at the same time it will be encouraged to resemble the result of the previous pass. Because the output of the filter is distorted by noise, the noise influence on the output of the filtering process will be accumulated. To alleviate such undesirable effects and to preserve such features as edges, it may be desirable to encourage the filter output to resemble the original signal. By observing the functional optimization property, we propose the following repeated recursive median filtering scheme

$$V_t^{(t)}(n) = \text{median}\left[V_i^{(t-1)}(n-N), \cdots, b_i(n), \cdots, V_i^{(t-1)}(n+N)\right] \tag{6}$$

where the superscript t is the iteration index and $V_i^{(0)}(n) = b_i(n)$.

This new algorithm can also be expressed by pseudo-C code as follows.

Algorithm 2

```
New Recursive-Median-Filter () {
  for (n = 1; n ≤ L; n++){V_i(n) = b_i(n);}
  do { success = 0;
      for (n = 1; n ≤ L; n++){
      m = median ((V_i(n − N), ···, b_i(n), ···, V_i(n + N))
      if (m == V_i(n)) success++
      V_i(n) = m; }}
  while (success ≠ L); }
```

That is, we use the original signal in the middle of the operation window throughout the whole process, instead of using the output of the previous pass. From the functional optimization properties of recursive median filtering, it can be easily understood that this operation has the properties of smoothing the signal and at the same time encourages the outputs of each pass to resemble the original (noisy) signal, instead of resembling the output of the previous pass that has been corrupted by the noise. Hopefully, in this way, such features as thin lines and sharp edges can be better preserved.

Property: Algorithm 2 converges within a finite number of iterations.

Proof: By induction, at each step of the algorithm, the following function is minimized:

$$E(V_i(n)) = -\sum_{\substack{j=-N \\ j \neq 0}}^{j=N} V_i(n) V_i(n+j) - V_i(n) b_i(n). \tag{7}$$

Because each point of the signal is sequentially visited by the recursive median filter and the output is updated before moving to the next position, we can easily show that the following function will be minimized by the process

$$E = -\frac{1}{2} \sum_{n=1}^{L} \sum_{\substack{j=-N \\ j \neq 0}}^{j=N} V_i(n) V_i(n+j) - \sum_{n=1}^{L} V_i(n) b_i(n) \tag{8}$$

where L is the total number of points of the signal. Because the output $V_i(n)$ is updated sequentially and only one output changes value at any time constant, the changes of the cost function in (6) caused by the changes of $V_i(n)$ are as follows:

$$\Delta E = \Delta V_i(n) S(n) \tag{9}$$

From (3), it is straightforward to show that ΔE is always less than or equal to zero. So after a finite number of iterations, E will reach its minimum, any further passes will not change the values of $V_i(n)$, and the process has converged, i.e., the signal has been reduced to root. □

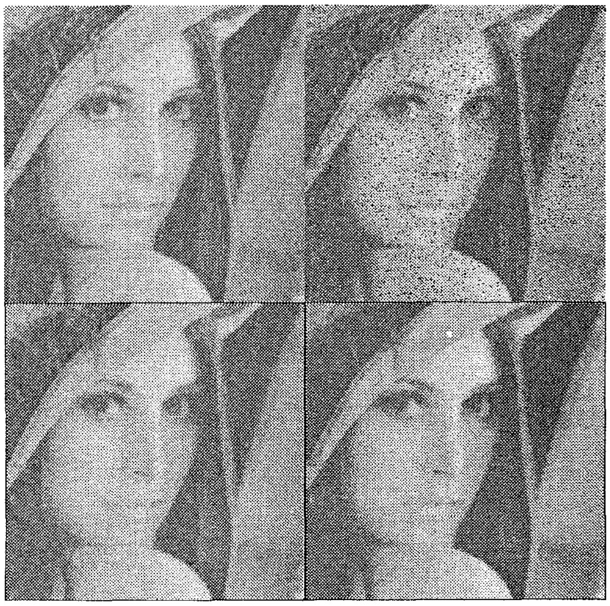

Fig. 1 Original image (upper left); noisy image (upper right); smoothed image using Algorithm 1 (lower left); smoothed image using Algorithm 2 (lower right).

TABLE I
COMPUTED MSE FOR FILTERED IMAGES CORRUPTED BY IMPULSIVE NOISE

METHOD	MSE
NOISY IMAGE	862.6
ALGORITHM ONE	41.4
ALGORITHM TWO	35.1

IV. SIMULATION RESULTS

To assess the performance of the new median filtering scheme, we applied it to a real image corrupted by pseudorandom computer-generated impulsive noise. The image considered contained 256×256 pixel values with eight bits resolution per pixel. The original image (uncorrupted) is shown in Fig. 1 (upper left). The noisy image is shown in Fig. 1 (upper right). Fig. 1 (lower left) shows the filtered image by using Algorithm 1. Fig. 1 (lower right) shows the filtered image by using Algorithm 2. In both cases, a 3×3 window was used and the threshold decomposition technique [6] was used in the simulation.

As can been seen, the differences in visual quality of the filtered images are slight. A useful quantitative comparison of the performance of the filtering schemes is the empirical mean square error, given by

$$\text{MSE} = \frac{1}{L}\sum_{j=1}^{L}(a(j) - y(j))^2 \tag{10}$$

where L is the total number of pixels in the image, $a(j)$ are the pixel values in the original image, $y(j)$ are the pixel values in the filtered image. Table I lists the empirical MSE for the filtered image using Algorithms 1 and 2, respectively. It is seen that the new method (Algorithm 2) provides better performance giving a MSE value reduced by 15.3%.

V. CONCLUDING REMARKS

In this paper, a new recursive median filtering scheme for image processing has been introduced. We prove that the new method converges in a finite number of iterations. Simulation results showed that the new scheme improves the MSE performance over the traditional method.

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Adaptive Restoration of Textured Images with Mixed Spectra

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***Abstract*— We consider the adaptive restoration of inhomogeneous textured images, where the individual regions are modeled using a Wold-like decomposition. A generalized Wiener filter is developed to accommodate mixed spectra, and unsupervised restoration is achieved by using the expectation-maximization (EM) algorithm to estimate the degradation parameters. This algorithm yields superior results when compared with supervised Wiener filtering using autoregressive (AR) image models.**

I. INTRODUCTION

Many natural images can be described as a collection of patches of fairly uniform textures. In this correspondence, we consider the

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